A Student's Guide to Maxwell's Equations

DANIEL FLEISCH
Wittenberg University

CAMBRIDGE
UNIVERSITY PRESS

$\boxed{\vec{\nabla}()}$ The gradient

To understand how Maxwell's Equations lead to the wave equation, it is necessary to comprehend a third differential operation used in vector calculus – the gradient. Similar to the divergence and the curl, the gradient involves partial derivatives taken in three orthogonal directions. However, whereas the divergence measures the tendency of a vector field to flow away from a point and the curl indicates the circulation of a vector field around a point, the gradient applies to *scalar fields*. Unlike a vector field, a scalar field is specified entirely by its magnitude at various locations: one example of a scalar field is the height of terrain above sea level.

What does the gradient tell you about a scalar field? Two important things: the magnitude of the gradient indicates how quickly the field is changing over space, and the direction of the gradient indicates the direction in that the field is changing most quickly with distance.

Therefore, although the gradient operates on a scalar field, the result of the gradient operation is a vector, with both magnitude and direction. Thus, if the scalar field represents terrain height, the magnitude of the gradient at any location tells you how steeply the ground is sloped at that location, and the direction of the gradient points *uphill* along the steepest slope.

The definition of the gradient of the scalar field ψ is

$$\mathrm{grad}(\psi) = \vec{\nabla}\psi \equiv \hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z} \qquad \text{(Cartesian)}. \tag{5.3}$$

Thus, the x-component of the gradient of ψ indicates the slope of the scalar field in the x-direction, the y-component indicates the slope in the y-direction, and the z-component indicates the slope in the z-direction. The square root of the sum of the squares of these components provides the total steepness of the slope at the location at which the gradient is taken.

In cylindrical and spherical coordinates, the gradient is

$$\vec{\nabla}\psi \equiv \hat{r}\frac{\partial\psi}{\partial r} + \hat{\varphi}\frac{1}{r}\frac{\partial\psi}{\partial\varphi} + \hat{z}\frac{\partial\psi}{\partial z} \qquad \text{(cylindrical)} \tag{5.4}$$

and

$$\vec{\nabla}\psi \equiv \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi} \qquad \text{(spherical)}. \tag{5.5}$$

$\vec{\nabla}, \vec{\nabla}\circ, \vec{\nabla}\times$ Some useful identities

Here is a quick review of the del differential operator and its three uses relevant to Maxwell's Equations:

Del:

$$\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Del (nabla) represents a multipurpose differential operator that can operate on scalar or vector fields and produce scalar or vector results.

Gradient:

$$\vec{\nabla}\psi \equiv \hat{i}\frac{\partial \psi}{\partial x} + \hat{j}\frac{\partial \psi}{\partial y} + \hat{k}\frac{\partial \psi}{\partial z}$$

The gradient operates on a scalar field and produces a vector result that indicates the rate of spatial change of the field at a point and the direction of steepest increase from that point.

Divergence:

$$\vec{\nabla} \circ \vec{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The divergence operates on a vector field and produces a scalar result that indicates the tendency of the field to flow away from a point.

Curl:

$$\vec{\nabla} \times \vec{A} \equiv \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

The curl operates on a vector field and produces a vector result that indicates the tendency of the field to circulate around a point and the direction of the axis of greatest circulation.

Once you're comfortable with the meaning of each of these operators, you should be aware of several useful relations between them (note that the following relations apply to fields that are continuous and that have continuous derivatives).

The curl of the gradient of any scalar field is zero.

$$\vec{\nabla} \times \vec{\nabla}\psi = 0, \tag{5.6}$$

which you may readily verify by taking the appropriate derivatives.

Another useful relation involves the divergence of the gradient of a scalar field; this is called the Laplacian of the field:

$$\vec{\nabla} \circ \vec{\nabla}\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \quad \text{(Cartesian)}. \tag{5.7}$$

The usefulness of these relations can be illustrated by applying them to the electric field as described by Maxwell's Equations. Consider, for example, the fact that the curl of the electrostatic field is zero (since electric field lines diverge from positive charge and converge upon negative charge, but do not circulate back upon themselves). Equation 5.6 indicates that as a curl-free (irrotational) field, the electrostatic field $\vec{E}$ may be treated as the gradient of another quantity called the scalar potential V:

$$\vec{E} = -\vec{\nabla}V, \tag{5.8}$$

where the minus sign is needed because the gradient points toward the greatest *increase* in the scalar field, and by convention the electric force on a positive charge is toward *lower* potential. Now apply the differential form of Gauss's law for electric fields:

$$\vec{\nabla} \circ \vec{E} = \frac{\rho}{\varepsilon_0},$$

which, combined with Equation 5.8, gives

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}. \tag{5.9}$$

This is called Laplace's equation, and it is often the best way to find the electrostatic field when you are not able to construct a special Gaussian surface. In such cases, it may be possible to solve Laplace's Equation for the electric potential V and then determine $\vec{E}$ by taking the gradient of the potential.

Here is a summary of the integral and differential forms of all of Maxwell's Equations in matter:

Gauss's law for electric fields:

$$\oint_S \vec{D} \circ \hat{n}\, da = q_{\text{free, enc}} \qquad \text{(integral form)},$$

$$\vec{\nabla} \circ \vec{D} = \rho_{\text{free}} \qquad \text{(differential form)}.$$

Gauss's law for magnetic fields:

$$\oint_S \vec{B} \circ \hat{n}\, da = 0 \qquad \text{(integral form)},$$

$$\vec{\nabla} \circ \vec{B} = 0 \qquad \text{(differential form)}.$$

Faraday's law:

$$\oint_C \vec{E} \circ d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \circ \hat{n}\, da \qquad \text{(integral form)},$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{(differential form)}.$$

Ampere–Maxwell law:

$$\oint_C \vec{H} \circ d\vec{l} = I_{\text{free, enc}} + \frac{d}{dt}\int_S \vec{D} \circ \hat{n}\, da \qquad \text{(integral form)},$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_{\text{free}} + \frac{\partial \vec{D}}{\partial t} \qquad \text{(differential form)}.$$

- **Divergence theorem**

Extending

$$\begin{aligned}\oint_S \mathbf{E}\cdot d\mathbf{s} &= \left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)\Delta x\,\Delta y\,\Delta z \\ &= (\text{div } \mathbf{E})\Delta\nu \end{aligned} \tag{140}$$

from differential volume $\Delta\nu$ to a volume integral,

$$\int_\nu \nabla\cdot\mathbf{E}d\nu = \oint_S \mathbf{E}\cdot d\mathbf{s} \quad \text{(divergence theorem)} \tag{141}$$

which is known as the divergence theorem . The closed surface in the integral on the right is the surface that bounds the volume that is integrated over on the left.

- **Vector identities involving curl**

$$(1) \quad \nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \qquad (149)$$

$$(2) \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad \text{for any vector } \mathbf{A} \qquad (150)$$

$$(3) \quad \nabla \times (\nabla V) = 0 \quad \text{for any scalar function V} \qquad (151)$$

- **Stoke's theorem**

Using this theorem we can convert the surface integral of the curl of a vector over an open surface S into a line integral of the vector along the contour C bounding the surface S.

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l} \quad \text{(Stokes's theorem)} \qquad (152)$$

If $\nabla \times \mathbf{B} = 0$ the field is said to be conservative or irrotational