Contents

5. Magnetostatics

For magnetostatics, we will build on our knowledge of electrostatics and explore some analogies between two.

Recall Maxwell's equations, obtained for steady-state,

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J} \tag{1}$$

where $\mathbf{J}$ is current density, $\mathbf{B}$ is magnetic flux density, and $\mathbf{H}$ is magnetic field intensity. They are related by

$$\mathbf{B} = \mu \mathbf{H} \tag{2}$$

For simplicity, assume that μ is scalar, linear and isotropic. For most dielectrics and metals $\mu = \mu_0$.

Table 5.1 in textbook has various relationships in electro- and magneto-statics cases.

5.1. Magnetic forces and torques

We've seen the relationship between electric field **E** and electric force $\mathbf{F}_\mathrm{e}$(what is it?). Magnetic flux density **B** is defined at a point in space as the **magnetic force F** that would be exerted on a charged particle moving with a velocity **u** at that point. This relationship is described by

$$\mathbf{F}_\mathrm{m} = q\mathbf{u} \times \mathbf{B} \quad \text{(N)} \tag{3}$$

- What is the unit for **B**? Additional unit introduced for **B** $\Rightarrow$ Tesla.
- For a positive charge $\Rightarrow$ force in the direction of $\mathbf{u} \times \mathbf{B}$ (which is where?); for a negative charge - opposite direction. Remember the right-hand rule for cross products. see Fig. 1.

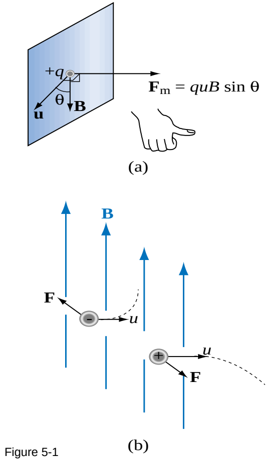


Figure 1: Direction of the magnetic force on a charged particle is: (a) perpendicular to both **B** and **u**, and (b) depends on charge polarity.

- The magnitude of the force is given by,

$$F_{\text{m}} = quB \sin\theta \tag{4}$$

where θ is angle between $\mathbf{u}$ and $\mathbf{B}$. What is the maximum and minimum?

- What if both $\mathbf{E}$ and $\mathbf{B}$ are present? Then we get both, **electromagnetic force**

$$\mathbf{F} = \mathbf{F}_{\text{e}} + \mathbf{F}_{\text{m}} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{5}$$

This is known as the Lorentz force.

- What are the differences between these two forces?

  1. $\mathbf{F}_{\rm e}$ is always in direction of electric field, but $\mathbf{F}_{\rm m}$ is always perpendicular to the magnetic field
  2. $\mathbf{F}_{\rm m}$ acts only on a moving charged particle; $\mathbf{F}_{\rm e}$ doesn't care.
  3. $\mathbf{F}_{\rm e}$ expends energy in displacing a charged particle; $\mathbf{F}_{\rm m}$ does no work, even though the particle is displaced.

- Remember that $\mathbf{F}_{\rm m}$ is perpendicular to $\mathbf{u} \Rightarrow \; \mathbf{F} \cdot \mathbf{u} =?$.

- The resulting work done when the particle is displaced along $d\mathbf{l} = \mathbf{u}dt$

$$dW = \mathbf{F}_m \cdot d\mathbf{l} = (\mathbf{F}_m \cdot \mathbf{u})\, dt = 0 \tag{6}$$

- Important: no work is done $\Rightarrow$ the magnetic field cannot change kinetic energy of a charged particle, but it can change the direction!

• Magnetic force on a current-carrying conductor

We can look at not just an isolated charged particle in motion, but also at a continuous flow e.g., current flowing through a wire. From above, if a current carrying wire is placed in a magnetic field, a force will act on the wire. Check Fig. 2 for the arrangement.

- For zero current - nothing happens and wire is vertical.
- For current flowing upward there is deflection in $-\mathbf{y}$ direction.
- For current flowing downward the deflection is in $\mathbf{y}$ direction.
- Can we quantify this? Recall that $\mathbf{F}_{\mathrm{m}} = q\mathbf{u} \times \mathbf{B}$ requires velocity $\mathbf{u}$, and we have current I.
- Take a small segment of the wire with cross-section of A and differential length $d\mathbf{l}$ where direction of $d\mathbf{l}$ is in the direction of the current I.
- Only dealing with electrons (conductor) $\Rightarrow$ charge density $\rho_{\mathrm{ve}} = -N_{\mathrm{e}}e$ where N_{e} is the number of moving electrons per unit volume.
- The total amount of moving charge

$$dQ = \rho_{\mathrm{ve}} A\, dl = -N_{\mathrm{e}} e A\, dl \tag{7}$$

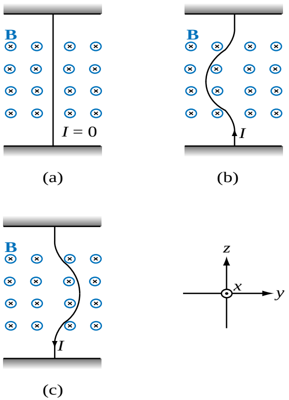


Figure 2: Wire in a magnetic field directed into the page: (a) no deflection when current through wire is zero, (b) deflection to the left when I is upward, and (c) deflection to the right when I is downward.

- The magnetic force acts on this charge

$$d\mathbf{F}_{\mathrm{m}} = dQ\,\mathbf{u}_{\mathrm{e}} \times \mathbf{B} = -N_{\mathrm{e}} e A\, dl\, \mathbf{u}_{\mathrm{e}} \times \mathbf{B} \tag{8}$$

- Note that the electron current flows in the direction opposite to the direction of their velocity, so that $dl\mathbf{u}_{\mathrm{e}} = -d\mathbf{l}u_{\mathrm{e}}$ and

$$d\mathbf{F}_{\mathrm{m}} = N_{\mathrm{e}} e A u_{\mathrm{e}}\, d\mathbf{l} \times \mathbf{B} \tag{9}$$

- How is the current calculated? We have the density, volume and velocity $\Rightarrow I = \rho_{\mathrm{ve}}(-u_{\mathrm{e}})A = (-N_{\mathrm{e}}e)(-u_{\mathrm{e}})A = N_{\mathrm{e}}eAu_{\mathrm{e}} \Rightarrow$

$$d\mathbf{F}_{\mathrm{m}} = I\, d\mathbf{l} \times \mathbf{B} \quad \text{(N)} \tag{10}$$

- If current I is flowing on a closed contour C, the total magnetic force becomes

$$\mathbf{F}_{\mathrm{m}} = I \oint_C d\mathbf{l} \times \mathbf{B} \quad \text{(N)} \tag{11}$$

Closed Circuit in a Uniform **B** Field

Check out Fig. 3; **B** is uniform and constant ⇒ take it out of integration

$$\mathbf{F}_m = I\left(\oint_C d\mathbf{l}\right) \times \mathbf{B} = 0 \tag{12}$$

which is statement of a geometrical fact that integral of $d\mathbf{l}$ over a closed path adds up to zero ⇒ total magnetic force on any closed current loop in a uniform magnetic field is zero.

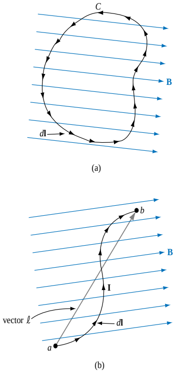


Figure 3: In a uniform **B**, (a) net force on a closed current loop is zero, and (b) force on a line segment is proportional to the vector between the end points ($\mathbf{F}_m = I\ell \times \mathbf{B}$).

Curved Wire in a Uniform **B** Field

Replacing the closed loop with wire segment in Fig. 3 and integrating gives:

$$\mathbf{F}_{\mathrm{m}} = I \left(\int_a^b d\mathbf{l} \right) \times \mathbf{B} = I\ell \times \mathbf{B} \tag{13}$$

where $\mathbf{l}$ is a vector directed from a to b. Integration along any path depends only on the start and final points, so use the simplest!

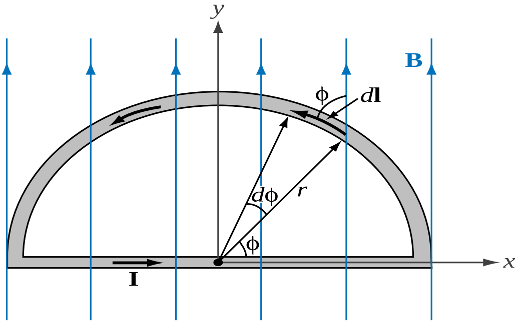


Figure 4: Semicircular conductor in a uniform field (Example 5-1).

• Magnetic torque on a current-carrying loop

Application of force on a rigid body that is pivoted about a fixed axis produces rotation about that axis. The "strength" of this rotation depends on the cross product of applied force **F** and the distance vector **d** which is measured from a point on the rotation axis to the point of application of force, as shown in Fig. 5.

$$\mathbf{T} = \mathbf{d} \times \mathbf{F} \quad (\text{N}\cdot\text{m}) \tag{14}$$

where **T** is **torque** and **d** is **moment arm**.

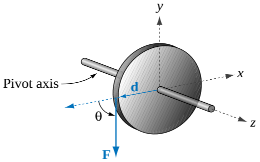


Figure 5: The force **F** acting on a circular disk pivoted along the z-axis generates a torque $\mathbf{T} = \mathbf{d} \times \mathbf{F}$ that causes the disk to rotate.

- Note that torque does not represent energy or work.
- For example in Fig. 5 the force lies in the $x - y$ plane and has angle θ with $\mathbf{d}$ $\Rightarrow$

$$\mathbf{T} = \hat{\mathbf{z}} r\, F\, \sin\theta \tag{15}$$

  where $|\mathbf{d}| = r$.

- Torque along $+z$ corresponds to rotation in CCW direction.

- Torque along $-z$ corresponds to rotation in CW direction.
- right-hand rule: thumb pointing along the direction of torque $\Rightarrow$ four fingers indicate direction in which torque is trying to rotate the body.
- Magnetic force, or magnetic flux density $\mathbf{B}$, can also produce torque.

Magnetic field in the plane of the loop

Start with Fig. 6: a rectangular conducting loop is made of rigid wire carrying current I. The pivot axis is shown and the loop lies in $x-y$ plane. The magnetic field (flux density) $\mathbf{B} = \hat{\mathbf{x}} B_0$.

- Divide the analysis into different sections of the loop. Take arms 1 and 3 first:

$$\mathbf{F}_1 = I(-\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = \hat{\mathbf{z}}IbB_0 \tag{16}$$

$$\mathbf{F}_3 = I(\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = -\hat{\mathbf{z}}IbB_0 \tag{17}$$

  where we've used $\mathbf{F}_{\mathrm{m}} = I\ell \times \mathbf{B}$.

- What about arms 2 and 4? They are parallel to $\mathbf{B} \Rightarrow 0$ force.

- Fig. 6 shows the forces and moment arms; where are the torques? What kind of rotation will this produce?

- What's the total torque? Add the two together; note that moment arms are the same in magnitude but opposite in direction:

$$\begin{aligned} \mathbf{T} &= \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3 \\ &= \left(-\hat{\mathbf{x}}\frac{a}{2}\right) \times (\hat{\mathbf{z}}IbB_0) + \left(\hat{\mathbf{x}}\frac{a}{2}\right) \times (-\hat{\mathbf{z}}IbB_0) \\ &= \hat{\mathbf{y}}IabB_0 = \hat{\mathbf{y}}IAB_0 \end{aligned} \tag{18}$$

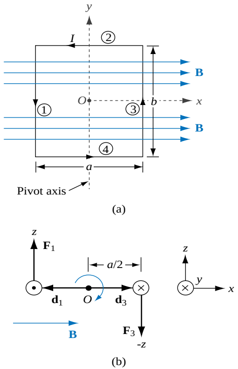


Figure 6: Rectangular loop pivoted along the y-axis; (a) front view and (b) bottom view. The combination of forces $\mathbf{F}_1$ and $\mathbf{F}_3$ on the loop generates a torque that tends to rotate the loop in a clockwise direction as shown in (b).

where $A = ab$ is area of the loop.

- Note that this result is valid only when $\mathbf{B}$ is parallel to the plane of the loop. Once the loop starts rotation $\Rightarrow$ $\mathbf{T}$ starts decreasing.
- What happens when they are perpendicular?

B perpendicular to the axis of a rectangular loop

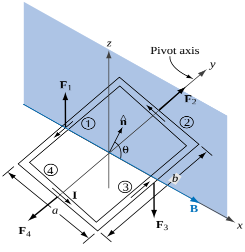
This setup is shown in Fig. 7 where $\mathbf{B} = \hat{\mathbf{x}}B_0$ is perpendicular to the axis of rotation but may be at an angle with the loop's surface normal $\hat{\mathbf{n}}$.

- Forces on arms 2 and 4 are non-zero, but what about their direction? $\Rightarrow$ net effect is zero force along the axis of rotation $\Rightarrow$ torque also zero.

- Direction of currents in arms 1 and 3 is always perpendicular to $\mathbf{B}$ regardless of the angle $\theta \Rightarrow$ forces will have the same expressions as before ($\mathbf{F}_1 = \hat{\mathbf{z}}IbB_0$ and $\mathbf{F}_3 = -\hat{\mathbf{z}}IbB_0$).

- What is different is that the cross product between the moment arm and force will have a $\sin\theta$ term, as shown in Fig. 7.

- Magnitude of the net torque is the same as before, ($\mathbf{T} = \hat{\mathbf{y}}IAB_0$) but with $\sin\theta$:
$$T = IAB_0 \sin\theta \tag{19}$$



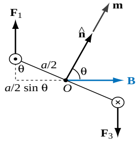


Figure 7: Rectangular loop in a uniform **B** whose direction is perpendicular to the rotation axis of the loop, but makes an angle θ with the loop's surface normal $\hat{\mathbf{n}}$.

- Where are the maxima and minima of this torque?
- If there are N loops (turns), then the total torque is obtained by multiplication:

$$T = NIAB_0 \sin\theta \tag{20}$$

- NIA is called **magnetic moment** m of the loop and can be viewed as a vector $\mathbf{m}$ with direction $\hat{\mathbf{n}}$ (surface normal).
- $\hat{\mathbf{n}}$ follows another right-hand rule: when the four fingers point in the direction of the current around the loop, the thumbs gives the direction of $\hat{\mathbf{n}}$.
- This gives magnetic moment:

$$\mathbf{m} \triangleq \hat{\mathbf{n}} NIA \quad (\text{A}\cdot\text{m}^2) \tag{21}$$

- The torque can now be expressed as,

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \quad (\text{N}\cdot\text{m}) \tag{22}$$

which is valid in general, not just for this orientation of **B** and this loop.

5.2. Biot-Savart Law

Let's look at a different problem: what is the magnetic field generated by a current? Also, we'll switch to magnetic field intensity $\mathbf{H}$ instead of magnetic flux density $\mathbf{B}$; remember that $\mathbf{B} = \mu\mathbf{H}$.

- An initial observation was made by Oersted: the deflection of compass needles by current flow in wires.

- Follow on: Jean Biot and Felix Savart derived expression relating $\mathbf{H}$ at any point in space to the current I that generates it $\Rightarrow$ Biot-Savart law:

$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \quad \text{(A/m)} \tag{23}$$

  where $d\mathbf{H}$ is generated by I flowing through differential length $d\mathbf{l}$. Vector $\mathbf{R} = \hat{\mathbf{R}}R$ is a distance vector between $d\mathbf{l}$ and the point of interest P (see Fig. 8).

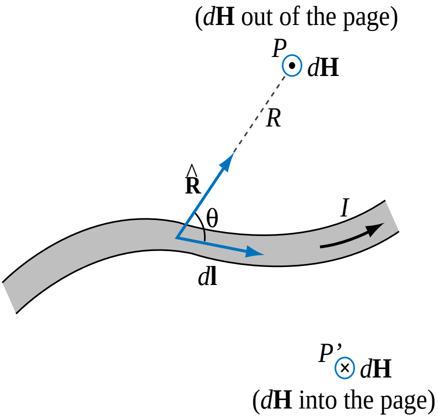


Figure 8: Magnetic field $d\mathbf{H}$ generated by current element $I\,d\mathbf{l}$. The direction of the field induced at point P is opposite that induced at point P’.

- What is the SI unit of $\mathbf{H}$? Amps/meter (A/m)
- Remember the reference directions: $d\mathbf{l}$ is in the direction of I and $\hat{\mathbf{R}}$ is from current element to P.
- Similarity as we had with $\mathbf{E}$ induced by charge: the fall-off goes as $\frac{1}{R^2}$.
- The difference: $\mathbf{E}$ is in the direction of the distance vector $\mathbf{R}$, while $\mathbf{H}$ is orthogonal to the plane defined by $d\mathbf{l}$ and $\mathbf{R}$.
- The direction of $\mathbf{H}$ is given in Fig. 8; note that at "opposite" points, field is in the opposite direction.
- This gives us the differential value of $\mathbf{H}$ at point P; what's the total value? $\Rightarrow$ sum up the contributions, i.e. integrate:

$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \quad \text{(A/m)} \tag{24}$$

where integration is along the path l where current I flows.

• Magnetic field due to current distributions

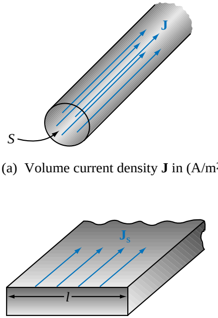
So far, we looked at a simple case of current in a wire. What happens if there is some distribution, volume or surface, of the current **density**?

- Surface current density shows up in cases where current flows in sheets of near-zero thickness. Fig. 9 shows two cases.
- Integrating volume current density (measured in A/m^2) over conductor **surface** gives the total current, i.e.

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \tag{25}$$

- Integrating surface current density (in A/m) over the conductor **length** gives total current,

$$I = \int_l J_s dl \tag{26}$$



(a) Volume current density **J** in (A/m²)

(b) Surface current density $\mathbf{J}_s$ in (A/m)

Figure 9: (a) The total current crossing the cross section S of the cylinder is $I = \int_S \mathbf{J} \cdot d\mathbf{s}$. (b) The total current flowing across the surface of the conductor is $I = \int_l J_S \, dl$.

- Note that the names are a bit confusing, but for our purpose here we can see that

$$I\,d\mathbf{l} = \mathbf{J}_s\,ds = \mathbf{J}\,dv \tag{27}$$

- Final result for Biot-Savart law:

$$\mathbf{H} = \frac{1}{4\pi}\int_S \frac{\mathbf{J_s}\times\hat{\mathbf{R}}}{R^2}\,ds \quad \text{(for a surface current)} \tag{28}$$

$$\mathbf{H} = \frac{1}{4\pi}\int_v \frac{\mathbf{J}\times\hat{\mathbf{R}}}{R^2}\,dv \quad \text{(for a volume current)} \tag{29}$$

- **Example 5-2: Magnetic field of a linear conductor**

$$\mathbf{H} = \frac{I}{4\pi}\int_{z=-l/2}^{z=l/2} \frac{d\mathbf{l}\times\hat{\mathbf{R}}}{R^2} = \hat{\phi}\frac{I}{4\pi}\int_{-l/2}^{l/2}\frac{\sin\theta}{R^2}\,dz \tag{30}$$

Variable substitution (transformation)

$$R = r\,\csc\theta,\quad z = -r\cot\theta,\quad dz = r\csc^2\theta\,d\theta \tag{31}$$

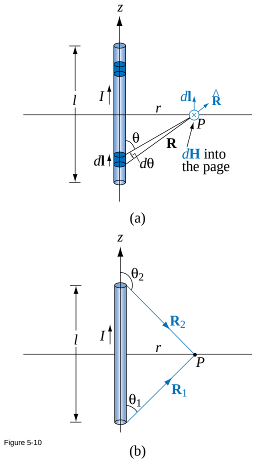


Figure 10: Linear conductor of length l carrying a current I. (a) $d\mathbf{H}$ at point P due to current element $d\mathbf{l}$. (b) Limiting angles θ_1 and θ_2.

$$\begin{aligned}\mathbf{H} &= \hat{\phi}\frac{I}{4\pi}\int_{\theta_1}^{\theta_2} \frac{\sin\theta\, r\csc^2\theta\, d\theta}{r^2\csc^2\theta} = \hat{\phi}\frac{I}{4\pi r}\int_{\theta_1}^{\theta_2}\sin\theta\, d\theta \\ &= \hat{\phi}\frac{I}{4\pi r}(\cos\theta_1 - \cos\theta_2) \qquad (32)\end{aligned}$$

$$\cos\theta_1 = \frac{l/2}{\sqrt{r^2+(l/2)^2}} \qquad (33)$$

$$\cos\theta_2 = -\cos\theta_1 = \frac{-l/2}{\sqrt{r^2+(l/2)^2}} \qquad (34)$$

$$\mathbf{B} = \mu_0\mathbf{H} = \hat{\phi}\frac{\mu_0 I l}{2\pi r\sqrt{4r^2+l^2}} \quad \text{(T)} \qquad (35)$$

$$\mathbf{B} = \hat{\phi}\frac{\mu_0 I}{2\pi r} \quad \text{(infinitely long wire)} \qquad (36)$$

- **Example 5-4: Magnetic field of a circular loop**

$$dH = \frac{I}{4\pi R^2}\,|d\mathbf{l} \times \hat{\mathbf{R}}| = \frac{I\,dl}{4\pi(a^2 + z^2)} \tag{37}$$

$$d\mathbf{H} = \hat{\mathbf{z}}\,dH_z = \hat{\mathbf{z}}\,dH\ \cos\theta = \hat{\mathbf{z}}\frac{I\cos\theta}{4\pi(a^2 + z^2)}\,dl \tag{38}$$

$$\mathbf{H} = \hat{\mathbf{z}}\frac{I\cos\theta}{4\pi(a^2 + z^2)}\oint dl = \hat{\mathbf{z}}\frac{I\cos\theta}{4\pi(a^2 + z^2)}(2\pi a) \tag{39}$$

$$\mathbf{H} = \hat{\mathbf{z}}\frac{Ia^2}{2(a^2 + z^2)^{3/2}} \quad \text{(A/m)} \tag{40}$$

$$\mathbf{H} = \hat{\mathbf{z}}\frac{I}{2a} \quad (\text{at } z = 0) \tag{41}$$

$$\mathbf{H} = \hat{\mathbf{z}}\frac{Ia^2}{2|z|^3} \quad (\text{at } |z| \gg a) \tag{42}$$

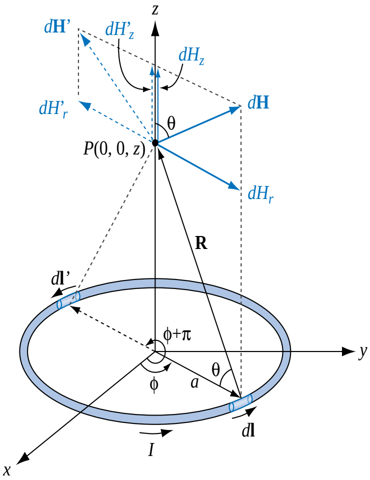


Figure 11: Circular loop carrying a current I (Example 5-4).

• Magnetic field of a magnetic dipole

Let's extend the analogy with the electric field by examining a magnetic dipole. The magnetic moment of a single current loop was introduced in $\mathbf{m} \triangleq \hat{\mathbf{n}}NIA \quad (\text{A}\cdot\text{m}^2)$.

- The loop is in the $x - y$ plane, as shown in Fig. 11.
- The magnetic moment of a current loop is in the z direction and has magnitude $m = I\pi a^2$ (why?) so that, $\mathbf{H} = \hat{\mathbf{z}}\frac{Ia^2}{2|z|^3}$ (eq. 5.36 in book) becomes:

$$\mathbf{H} = \hat{\mathbf{z}}\frac{m}{2\pi|z|^3} \quad (\text{at } |z| \gg a) \tag{43}$$

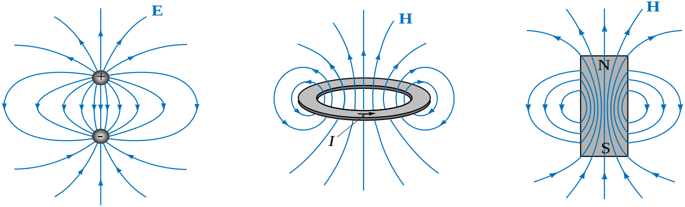
  which is OK if P is far away from the loop.

- For the same problem, but set up in spherical coordinates and distant point P' we get

$$\mathbf{H} = \frac{m}{4\pi R'^3}\left(\hat{\mathbf{R}}2\cos\theta' + \hat{\theta}\sin\theta'\right) \tag{44}$$

with condition $R' \gg a$.

- Such current loop with dimensions much smaller than R is called **magnetic dipole**.
- Rationale: magnetic field pattern looks similar to the pattern of electric dipole field — see Fig. 12



(a) Electric dipole (b) Magnetic dipole (c) Bar magnet

Figure 12: Field lines: (a) electric dipole, (b) magnetic dipole, and (c) bar magnet. Far away from the sources, the field patterns are similar.

5.3. Magnetic force between parallel conductors

In the previous examples we looked at current carrying wires in a constant magnetic field. But, the Biot-Savart law tells us that there is magnetic field due to current flow itself ⇒ a current carrying wire can exert force on another current carrying wire.

- Consider two very long wires, parallel to each other, as in Fig. 13, separated by d, carrying currents I_1, I_2 in the same direction.
- The coordinate system origin is set up in the middle with axis arrangement as shown.

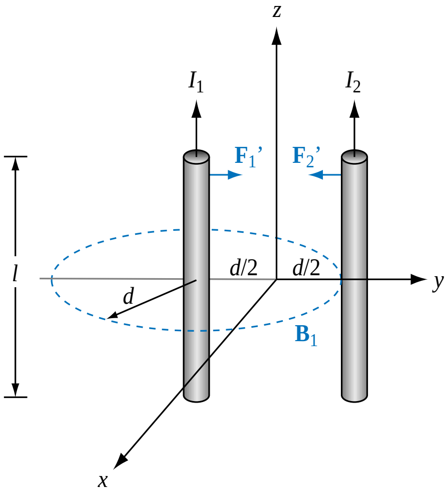


Figure 13: Magnetic forces on parallel current-carrying conductors.

- Denote $\mathbf{B}_1$ as the magnetic field due to current I_1 at the location of the second wire (I_2), and vice versa.

- Use $\mathbf{B} = \hat{\phi}\frac{\mu_0 I}{2\pi r}$ (note: $I = I_1$, $r = d$, $\hat{\phi} = -\hat{\mathbf{x}}$) to find $\mathbf{B}_1 \Rightarrow$

$$\mathbf{B}_1 = -\hat{\mathbf{x}}\frac{\mu_0 I_1}{2\pi d} \tag{45}$$

- How do we find the force? Integrate along the length l along the line:

$$\mathbf{F}_{\mathrm{m}} = I\left(\int_a^b d\mathbf{l}\right) \times \mathbf{B} = I\ell \times \mathbf{B} \tag{46}$$

$$\mathbf{F}_2 = I_2 l\hat{\mathbf{z}} \times \mathbf{B}_1 = I_2 l\hat{\mathbf{z}} \times (-\hat{\mathbf{x}})\frac{\mu_0 I_1}{2\pi d} = -\hat{\mathbf{y}}\frac{\mu_0 I_1 I_2 l}{2\pi d} \tag{47}$$

- or, to get force per unit length

$$\mathbf{F}_2' = \frac{\mathbf{F}_2}{l} = -\hat{\mathbf{y}}\frac{\mu_0 I_1 I_2}{2\pi d} \tag{48}$$

- Completely analogous analysis leads to

$$\mathbf{F}_1' = \hat{\mathbf{y}} \frac{\mu_0 I_1 I_2}{2\pi d} \tag{49}$$

i.e. $\mathbf{F}_1' = -\mathbf{F}_2' \Rightarrow$ the wires attract each other with equal forces.

- What happens if currents are in opposite directions?

5.4. Maxwell's Magnetostatic Equations

In this section we will look into two of Maxwell's equations, Gauss's law for magnetism and Ampere's law.

- **Gauss's law for magnetism**

Recall, in chapter 4 the discussion of Gauss's law:

$$\nabla \cdot \mathbf{D} = \rho_v \Longleftrightarrow \oint_S \mathbf{D} \cdot d\mathbf{s} = Q \tag{50}$$

Differential and integral form are equivalent and are obtained via *divergence theorem* which can be applied in general.

- Interpretation: the total charge inside the surface S is determined by the surface integral of $\mathbf{D} \cdot d\mathbf{s}$.

- Problem: magnetic poles come only in pairs! $\Rightarrow$ **Gauss's law for magnetism** becomes

$$\nabla \cdot \mathbf{B} = 0 \Longleftrightarrow \oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \tag{51}$$

- Interpretation: the electric field lines originate and terminate on charges; magnetic field lines form closed loops! See fig. 14.

- If there is some net charge inside the surface there are some lines that have to terminate on charges outside the volume. However, equal "amounts" of magnetic lines leave as enter the surface, explaining eq. 51.

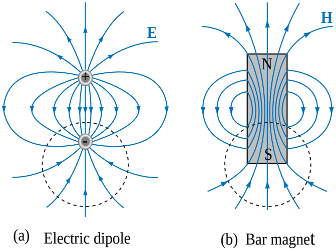


Figure 14: Whereas (a) the net electric flux through a closed surface surrounding a charge is not zero, (b) the net magnetic flux through a closed surface surrounding one of the poles of a magnet is zero.

• Ampere's law

Next, we look at another of Maxwell's equation for magnetostatics, Ampere's law:

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{52}$$

the integral form requires a bit of work:

- Integrate both sides over an open surface S,

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s} \tag{53}$$

- Recognize that the right side is just the current I
- Then use Stoke's theorem:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \oint_C \mathbf{H} \cdot d\mathbf{l} \quad \text{(Stokes's theorem)} \tag{54}$$

- To get,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I \quad \text{(Ampère's law)} \tag{55}$$

where C is closed contour bounding the surface S and $I = \int \mathbf{J} \cdot d\mathbf{s}$ is the current flowing through S.

- The direction of C is such that I and $\mathbf{H}$ satisfy the right-hand rule that was discussed in the Biot-Savart law.

- Ampere's circuital law: the line integral of $\mathbf{H}$ around a closed path is equal to the current traversing the surface bounded by that path.

- Several cases illustrated in Fig. 15. Note that the shape of the contour has no effect on the final result.

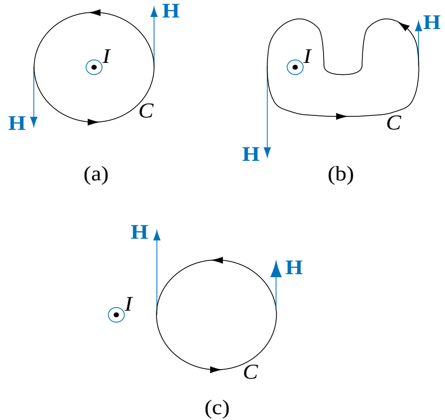


Figure 15: Illustration of Ampère's law: line integral of **H** around a closed contour C is equal to the current traversing the surface bounded by the contour.

- **Example 5-5: Magnetic field of a long wire**

$$\mathbf{H}_1 = \hat{\phi} H_1 = \hat{\phi} \frac{r_1}{2\pi a^2} I \quad (\text{for } r_1 \leq a) \tag{56}$$

$$\mathbf{H}_2 = \hat{\phi} H_2 = \hat{\phi} \frac{I}{2\pi r_2} \quad (\text{for } r_2 \geq a) \tag{57}$$

- **Example 5-6: Magnetic field inside a toroidal coil**

$$\mathbf{H} = -\hat{\phi} H = -\hat{\phi} \frac{NI}{2\pi r} \quad (\text{for } a < r < b) \tag{58}$$

- **Example 5-7: Magnetic field of an infinite current sheet**

$$\mathbf{H} = \begin{cases} -\hat{\mathbf{y}} \frac{\mathbf{J}_s}{2}, & \text{for } z > 0, \\ \hat{\mathbf{y}} \frac{\mathbf{J}_s}{2}, & \text{for } z < 0. \end{cases} \tag{59}$$

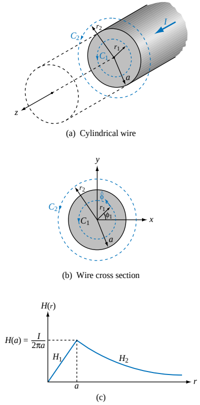


Figure 16: Infinitely long wire of radius a carrying a uniform current I along the $+z$-direction: (a) general configuration showing contours C_1 and C_2; (b) cross-sectional view; and (c) a plot of H versus r (Example 5-5).

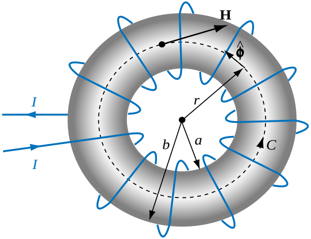


Figure 17: Toroidal coil with inner radius a and outer radius b. (Example 5-6).

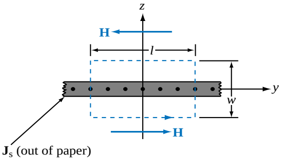


Figure 18: A thin current sheet in the x-y plane carrying a surface current density $\mathbf{J}_s = \hat{\mathbf{x}} J_s$ (out of the page) (Example 5-7).

5.5. Vector magnetic potential

Let's build on our experience with the electric field and potential: $\mathbf{E} = -\nabla V$. This is used to find electric field if potential is known, which can be simpler than dealing with electric field directly. How about magnetic flux density $\mathbf{B}$?

- We know from before that $\nabla \cdot \mathbf{B} = 0$ so we want to find the magnetic potential that will guarantee this.

- We also know that $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ (for any vector) $\Rightarrow$ defining vector magnetic potential $\mathbf{A}$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (\mathrm{Wb/m^2}) \tag{60}$$

guarantees the above condition.

- Units? Even though tesla (T) is SI unit for $\mathbf{B}$, another one is used: $\mathrm{Wb/m^2} \Rightarrow \mathbf{A} = \mathrm{Wb/m}$.

- Magnetic vector potential can be used to re-write some equations, e.g. Ampere's law (eq. 52)

$$\nabla \times \mathbf{B} = \mu \mathbf{J} \Rightarrow \nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{J} \tag{61}$$

- Recall that

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \tag{62}$$

and

$$\nabla^2 \mathbf{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{A}$$
$$= \hat{\mathbf{x}} \nabla^2 A_x + \hat{\mathbf{y}} \nabla^2 A_y + \hat{\mathbf{z}} \nabla^2 A_z \tag{63}$$

- Combining eqs. 61 and 62 gives

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} \tag{64}$$

- This can be simplified even further since we have a choice of functional form for $\nabla \cdot \mathbf{A} \Rightarrow$ make it zero!

- Finally, we get the *vector Poisson's equation*

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \tag{65}$$

which can be broken down into three scalar Poisson's equations:

$$\nabla^2 A_x = -\mu J_x, \quad \nabla^2 A_y = -\mu J_y, \quad \nabla^2 A_z = -\mu J_z \tag{66}$$

- In electrostatics the solution to Poisson's equation

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \quad \text{is} \quad V = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{\rho_v}{R'}\, dv' \tag{67}$$

- For magnetostatics similar solution, but instead of ρ/ε we have μJ:

$$A_x = \frac{\mu}{4\pi} \int_{v'} \frac{J_x}{R'}\, dv' \quad \text{(Wb/m)} \tag{68}$$

  and similarly for other components.

- Putting it all together into one *vector* equation:

$$\mathbf{A} = \frac{\mu}{4\pi} \int_{v'} \frac{\mathbf{J}}{R'}\, dv' \quad \text{(Wb/m)} \tag{69}$$

- If surface (or line) current density is known, then integration is over a surface S' (line l').

- Solutions via: Ampere's law, Biot-Savart law or magnetic potential; whichever is the easiest to use.

Introduce another quantity: magnetic flux Φ. Defined as total magnetic flux density passing through surface S:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \quad \text{(Wb)} \tag{70}$$

Use magnetic potential $\mathbf{B} = \nabla \times \mathbf{A}$ and Stoke's theorem to get,

$$\Phi = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l} \quad \text{(Wb)} \tag{71}$$

where C is the contour bounding the surface S.

5.6. Magnetic properties of materials

We'll use a classical picture of the atom to discuss magnetic properties. Recall that a loop carrying current generates magnetic field profile similar to permanent magnet ⇒ magnetic moment $m = IA$. There are two sources of magnetization in a material:

1. Orbital motions of electrons around the nucleus (and protons in nucleus)

2. Electron spin

Magnetic behavior is governed by interaction between magnetic dipole moments and external magnetic field. Materials are classified as:

1. diamagnetic — have no permanent magnetic dipole moments

2. paramagnetic — have dipole moments

3. ferromagnetic — have dipole moments (but different structure from paramagnetic)

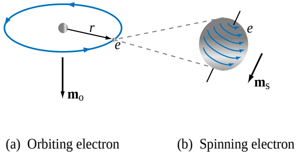


Figure 19: An electron generates (a) an orbital magnetic moment $\mathbf{m}_o$ as it rotates around the nucleus and (b) a spin magnetic moment $\mathbf{m}_s$, as it spins about its own axis.

- **Orbital and spin magnetic moments**

The general idea is given in Fig. 19:

- An electron orbiting with constant velocity and period $T = 2\pi r/u$ produces current,

$$I = -\frac{e}{T} = -\frac{eu}{2\pi r} \tag{72}$$

- This results in orbital magnetic moment $\mathbf{m}_0$

$$m_o = IA = \left(-\frac{eu}{2\pi r}\right)(\pi r^2) = -\frac{eur}{2} = -\left(\frac{e}{2m_e}\right) L_e \qquad (73)$$

where $L_e = m_e ur$ is the angular momentum.

- Quantum physics modifies this picture: L_e is quantized, $L_e = 0, \hbar, 2\hbar \ldots$. (where $\hbar = h/2\pi$ and h is Planck's constant)

- $\Rightarrow$ there is a minimum (non-zero) orbital magnetic moment

$$m_o = -\frac{e\hbar}{2m_e} \qquad (74)$$

- Most materials are nonmagnetic. Atoms are oriented randomly so that magnetic moments of electrons add up to zero or small value.

- What about the spin? Electron generates its own **spin magnetic moment $\mathbf{m}_s$** (Fig. 19). From quantum theory

$$m_s = -\frac{e\hbar}{2m_e} \tag{75}$$

which is the same as minimum orbital magnetic moment.

- For spin magnetic moment to show up we need odd number of electrons per atom. (They pair up in opposite directions so cancel if even.)

- Due to higher mass of the nucleus, it has much lower magnetic moment.

• Magnetic Permeability

Start by defining magnetization vector **M** as vector sum of all magnetic dipole moments.

- To get the total magnetic flux density inside the material, we have to add internal to external component:

$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{76}$$

- Magnetization happens in response to the external field (in analogy with electric field case), so that

$$\mathbf{M} = \chi_m \mathbf{H} \tag{77}$$

  where χ_m is **magnetic susceptibility** of the material.

- Diamagnetic and paramagnetic materials have constant χ_m but not ferromagnetic ones.

- Express everything on RHS in terms of **H**:

$$\mathbf{B} = \mu_0(\mathbf{H} + \chi_m \mathbf{H}) = \mu_0(1 + \chi_m)\mathbf{H} \tag{78}$$

- Use just one const. on RHS: **magnetic permeability** μ

$$\mathbf{B} = \mu \mathbf{H}, \quad \text{where} \quad \mu = \mu_0(1 + \chi_m) \quad \text{(H/m)} \tag{79}$$

- Define **relative permeability**

$$\mu_r = \frac{\mu}{\mu_0} = 1 + \chi_m \tag{80}$$

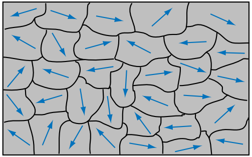
  where μ_0 is permeability of free space.

- Table 5-2 for classification of materials; based on value of χ_m.
- Most metals and dielectric materials have $\mu_r \approx 1$ (or $\mu = \mu_0$)
- Ferromagnetic materials have very large μ_r

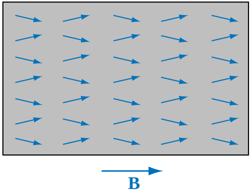
• Magnetic hysteresis of ferromagnetic materials

Special materials: iron, nickel, cobalt. Their magnetic moments align "easily" with external magnetic field. In addition, they retain their magnetization even after the external field is removed! Here is their qualitative description.

- Magnetized domains are essential feature of ferromagnetic materials, and are illustrated in Fig. 20. Within each (microscopic) domain moments of all its atoms are aligned.
- Without an external field, each domain's field orientation is random so that the net magnetization is zero.
- Under the influence of external field, domain magnetization will align (partially) with it (Fig. 20).
- **Magnetization curve** describes behavior of ferromangetics under an external field **H**. It shows values of internal magnetic flux density **B** for a given **H**.



(a) Unmagnetized domains



(b) Magnetized domains

Figure 20: Comparison of (a) unmagnetized and (b) magnetized domains in a ferromagnetic material.

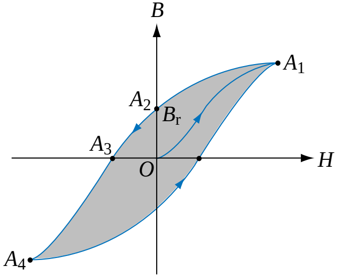


Figure 21: Typical hysteresis curve for a ferromagnetic material.

- Typical magnetization curve shown in Fig. 21. How do we explain it?
- Start with unmagnetized material (origin). As H increases, so does B (linearly?), until we start entering saturation.

- At point A_1 we reverse the process and start decreasing H. Interestingly, this does not follow the same curve as on the way "up." Even for zero H there is some **residual flux density** B_r. At this point material is magnetized and can be used as permanent magnet.

- Further increase in the opposite direction of H first leads to zero internal field and then to saturation at A_4.

- From A_4 reduction to zero still leaves some residual magnetization (but opposite to the one before). Finally, we reach A_1 saturation point.

- This characteristics is called magnetic hysteresis. Relationship between **H** and **B** is not unique and also depends on "history" of magnetization.

- Distinguish **hard** and **soft** ferromagnetic materials, shown in Fig. 22. Hard ones have wide hysteresis loop while soft ones

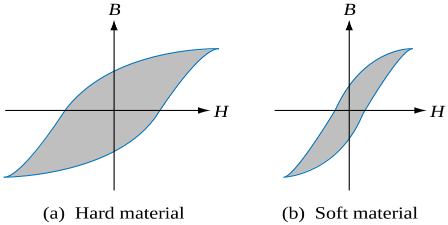


Figure 22: Hysteresis curves for (a) hard, and (b) soft ferromagnetic material.

have narrow loop. Hard ones are used for permanent magnets, while soft ones are easier to magnetize/demagnetize.

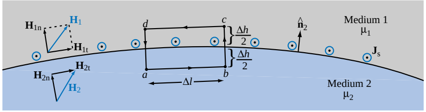


Figure 23: Boundary between medium 1 with μ_1 and medium 2 with μ_2.

5.7. Magnetic boundary conditions

We'll build on the foundation established for electrostatic boundary conditions. Figure 23 shows the boundary. First recall the Gauss's law at boundary

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \Longrightarrow D_{1n} - D_{2n} = \rho_s \tag{81}$$

- We have Gauss's law for magnetism (eq. 51) which produces

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \Longrightarrow B_{1n} = B_{2n} \tag{82}$$

i.e. **normal component of B is continuous across the boundary between two adjacent media**. This can be extended to magnetic field intensity:

$$\mu_1 H_{1n} = \mu_2 H_{2n} \tag{83}$$

- Difference: **B** is continuous across the boundary, but **D** is not, unless $\rho = 0$. What about tangential component?

- Apply Ampere's law around rectangular path and let $\Delta h \to 0$ so that

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_a^b \mathbf{H}_2 \cdot d\mathbf{l} + \int_c^d \mathbf{H}_1 \cdot d\mathbf{l} = I \tag{84}$$

- Notice that along ab and cd segments, tangential H-s are parallel to $d\mathbf{l}$ (what about sign?). As $\Delta h \to 0$, surface becomes a line of length Δl. What is the current on this line? $I = J_s \Delta l \Rightarrow$

$$H_{2t}\,\Delta l - H_{1t}\,\Delta l = J_s\,\Delta l \Rightarrow H_{2t} - H_{1t} = J_s \quad (\mathrm{A/m}) \tag{85}$$

- In a general vector form:

$$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \tag{86}$$

where $\hat{\mathbf{n}}_2$ is the normal unit vector pointing away from medium 2.

- Surface currents exist only on the surfaces of superconductors (or perfect conductors). For media with finite conductivities, at the interface we have $J_s = 0$ and

$$H_{1\,t} = H_{2t} \tag{87}$$

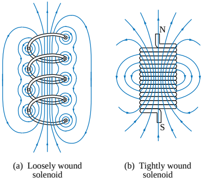


Figure 24: Magnetic field lines of a) loosely, and b) tightly wound solenoid.

5.8. Inductance

Electric energy is stored in a capacitor; magnetic energy in an inductor. An example is shown in Fig. 24: solenoid. Cores can be of different types of materials: air or magnetic with permeability μ. If wound tightly, the solenoid magnetic field will be similar to that of a permanent magnet.

• Magnetic field in a solenoid

The goal is to find **B** inside the solenoid. Figure 25 shows the arrangement.

- Treat the turns as circular loops of radius a, carrying current I.
- From section 5.2.3 we know **H** along the z axis:

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I' a^2}{2(a^2 + z^2)^{3/2}} \tag{88}$$

- Look at the dz length of the solenoid: there are $n dz$ turns with the total current $I' = I n dz \Rightarrow$

$$d\mathbf{B} = \mu \, d\mathbf{H} = \hat{\mathbf{z}} \frac{\mu n I a^2}{2(a^2 + z^2)^{3/2}} \, dz \tag{89}$$

- What is needed to get the total **B**? Some substitutions and integration ...

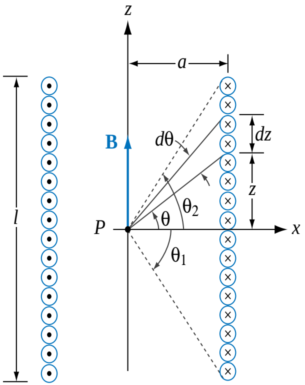


Figure 25: Solenoid cross section showing geometry for calculating **H** at a point P on the solenoid axis.

$$\mathbf{B} = \hat{\mathbf{z}}\frac{\mu n I a^2}{2}\int_{\theta_1}^{\theta_2} \frac{a \sec^2\theta\, d\theta}{a^3 \sec^3\theta} = \hat{\mathbf{z}}\frac{\mu n I}{2}(\sin\theta_2 - \sin\theta_1) \qquad (90)$$

- Approximation: $l \gg a \Rightarrow \theta_1 \approx -90^\circ$ and $\theta_2 \approx 90^\circ \Rightarrow$

$$\mathbf{B} \simeq \hat{\mathbf{z}}\mu n l = \frac{\hat{\mathbf{z}}\mu N I}{l} \quad \text{(long solenoid with } l/a \gg 1\text{)} \qquad (91)$$

  where $N = nl$ is total number of turns over the length l. Even though derived just for the midpoint this is approximately valid everywhere in the interior of the solenoid except near the ends.

Two types of inductance: **self-inductance** and **mutual inductance**. Self inductance represents magnetic flux linkage of a coil (or circuit) with itself, while mutual inductance involves magnetic flux linkage generated by a current in another coil (or circuit).

- **Self-inductance**

We've already seen that the magnetic flux Φ linking a surface S is given by

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \quad \text{(Wb)} \tag{92}$$

- How does this work out for solenoid? For a single loop we integrate over the cross-section of the solenoid (loop):

$$\Phi = \int_S \hat{\mathbf{z}} \left(\mu \frac{N}{l} I \right) \cdot \hat{\mathbf{z}}\, ds = \mu \frac{N}{l} IS \tag{93}$$

- Magnetic flux linkage Λ is defined as the total magnetic flux linking a given circuit or conducting structure.

- For solenoid, total Λ is simply

$$\Lambda = N\Phi = \mu \frac{N^2}{l} IS \quad \text{(Wb)} \tag{94}$$

- Things are a bit different for structures with separate conductors, as in parallel-wire or coax lines (Fig. 26). There the flux linkage Λ associated with a length l of line refers to the flux Φ through the surface *between* two conductors.

- Note that we've assumed zero magnetic field inside the lines.

- Define self-inductance as:

$$L = \frac{\Lambda}{I} \quad \text{(H)} \tag{95}$$

measured in henry (H) or (Wb/A).

- For a solenoid:

$$L = \mu \frac{N^2}{l} S \quad \text{(solenoid)} \tag{96}$$

- The two conductor configuration:

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s} \tag{97}$$

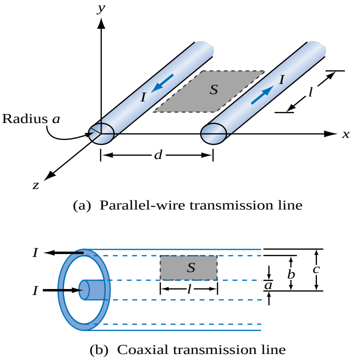


Figure 26: To compute the inductance per unit length of a two-conductor transmission line, we need to determine the magnetic flux through the area S between the conductors.

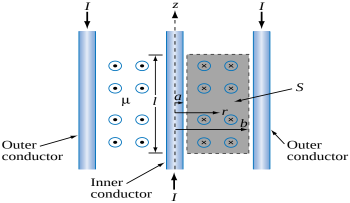


Figure 27: Cross-sectional view of coaxial transmission line (Ex. 5-8).

- **Example 5-8: Inductance of a coaxial transmission line**

Setup shown in Fig. 27. From before (eq. 5.30), current I in the inner conductor produces:

$$\mathbf{B} = \hat{\phi}\frac{\mu I}{2\pi r} \tag{98}$$

B is perpendicular to the surface everywhere:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \mathbf{B}_\phi \cdot \hat{\phi} dr dz = \int_0^l \int_a^b B \, dr dz \tag{99}$$

$$\Phi = l \int_a^b B \, dr = l \int_a^b \frac{\mu I}{2\pi r} \, dr = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right) \tag{100}$$

$$L' = \frac{L}{l} = \frac{\Phi}{lI} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) \tag{101}$$

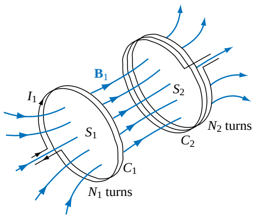


Figure 28: Magnetic field lines generated by current I_1 in loop 1 linking surface S_2 of loop 2.

- **Mutual inductance**

To describe the magnetic coupling between two circuits, we need mutual inductance. See Fig. 28 for a simple case of two loops.

- Current I_1 generates magnetic field $\mathbf{B}_1$ which results in a flux

Φ_{12} through loop 2:

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s} \tag{102}$$

- If loop 2 has N_2 turns that all couple to B_1 the same way then

$$\Lambda_{12} = N_2 \Phi_{12} = N_2 \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s} \tag{103}$$

- Finding mutual inductance is then easy:

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s} \quad \text{(H)} \tag{104}$$

- The primary use is in transformers (Fig. 3).

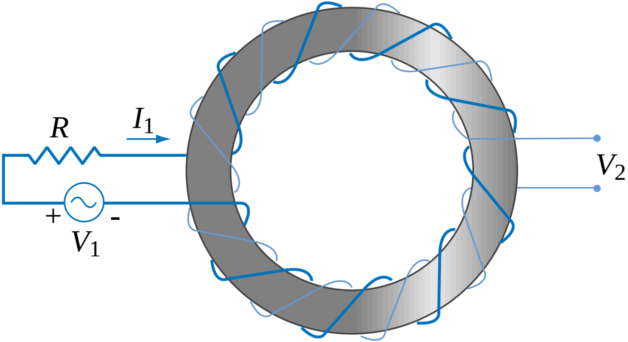


Figure 29: Toroidal coil with two windings used as a transformer.

5.9. Magnetic Energy

We'll use a simple approach, using what we know from circuit theory, i.e. $v = L di/dt$, so that the energy (integral of power) is

$$W_m = \int p\, dt = \int iv\, dt = L \int_0^I i\, di = \frac{1}{2} L I^2 \quad \text{(J)} \tag{105}$$

and this is our **magnetic energy** stored in the inductor.

- We know the inductance of the solenoid (eq. 96) as well as magnetic flux density inside it (eq. 91) ⇒

$$W_m = \frac{1}{2} L I^2 = \frac{1}{2} \left(\mu \frac{N^2}{l} S \right) \left(\frac{Bl}{\mu N} \right)^2 = \frac{1}{2} \frac{B^2}{\mu} (lS) = \frac{1}{2} \mu H^2 v \tag{106}$$

  where $v = lS$ is the volume of the solenoid and $H = B/\mu$.

- *Magnetic energy density* w_m is above equation per unit volume:

$$w_m = \frac{W_m}{v} = \frac{1}{2} \mu H^2 \quad (\text{J/m}^3) \tag{107}$$

which is valid for any medium with magnetic field **H**.

- Use energy density to find the total magnetic energy associated with **H**:

$$W_m = \frac{1}{2}\int_v \mu H^2 \, dv \quad \text{(J)} \tag{108}$$

- **Example 5-9: Magnetic energy in a coax cable**

Eq. 98 gives

$$H = \frac{B}{\mu} = \frac{I}{2\pi r} \tag{109}$$

(see fig. 27). Magnetic energy is then

$$W_m = \frac{1}{2}\int_v \mu H^2 dv = \frac{\mu I^2}{8\pi^2}\int_v \frac{1}{r^2} dv \tag{110}$$

Choose dv to be a cylindrical shell of length l, radius r and thickness $dr \Rightarrow dv = 2\pi r l dr$

$$W_m = \frac{\mu I^2}{8\pi^2} \int_a^b \frac{1}{r^2} 2\pi r l dr = \frac{\mu I^2 l}{4\pi} \ln\left(\frac{b}{a}\right) \text{ (J)} \qquad (111)$$