Contents

4. Electrostatics

4.1. Maxwell's equations

The starting point for our discussion of electromagnetics is Maxwell's equations:

$$\nabla \cdot \mathbf{D} = \rho_v \quad (1)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (4)$$

where the field quantities are related to each other: $\mathbf{D} = \epsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$. New quantities introduced: electric charge density ρ_{v}, in $\mathrm{C/m^3}$ and $\mathbf{J}$ which is current density ($\mathrm{A/m^2}$).

The **static** case simplifies these expressions, i.e. when charges are not moving or moving with constant rate $\Rightarrow$ ρ_v and $\mathbf{J}$ are constant in time (time-derivatives are zero) $\Rightarrow$

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_v \tag{5}$$
$$\nabla \times \mathbf{E} = 0 \tag{6}$$

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0 \tag{7}$$
$$\nabla \times \mathbf{H} = \mathbf{J} \tag{8}$$

Why is this simpler than the original equations? No time-derivatives and the electric and magnetic fields are no longer coupled! (Remember: this is only for static cases). Despite its "simplicity" solutions of these equations are very useful.

4.2. Charge and current distributions

If charge is moving $\Rightarrow$ current flows. Charges can have various distributions: over a volume of space, across a surface or along a line.

- **Charge density**

If we can neglect the microscopic, discrete, picture, then we talk about average quantities, such as net charge contained in some volume. If the volume is small, assume uniform distribution,

- For volume charge density we have

$$\rho_{\mathrm{v}} = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv} \quad (\mathrm{C/m^3}) \tag{9}$$

This charge density can vary from point to point, so we talk of *spatial distribution* of charge. To get the total charge, integration (summation) is needed:

$$Q = \int_v \rho_\mathrm{v}\, dv \quad \text{(C)} \tag{10}$$

- Surface charge density - charge distributed over a surface so that

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \quad (\mathrm{C/m^2}) \tag{11}$$

  where Δq is the charge on a small surface element Δs.

- Line charge density - charge distributed along a line

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad (\mathrm{C/m}) \tag{12}$$

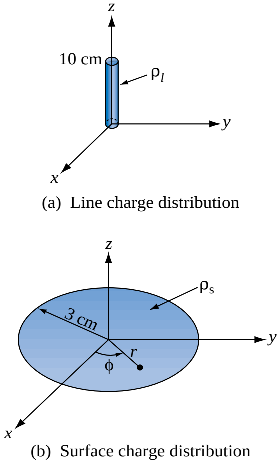


Figure 1: Charge distributions for Examples 4-1 and 4-2.

- **Current density**

Consider charges moving inside a "tube" (cylinder) in Fig. 2. Charge moves with average velocity $\mathbf{u}$ in the direction indicated. In Δt they move $\Delta l = u\Delta t$.

How many charges cross the cross-section $\Delta s'$ in Δt?

$$\Delta q' = \rho_{\mathrm{v}}\,\Delta v = \rho_{\mathrm{v}}\,\Delta l\,\Delta s' = \rho_{\mathrm{v}} u\,\Delta s'\,\Delta t \tag{13}$$

More generally, flow is not perpendicular to the surface, so we have to look at the projection of flow onto the surface normal $\hat{\mathbf{n}}$.

$$\Delta q = \rho_{\mathrm{v}}\mathbf{u}\cdot\Delta\mathbf{s}\,\Delta t \tag{14}$$

From this, finding current is easy

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_{\mathrm{v}}\mathbf{u}\cdot\Delta\mathbf{s} = \mathbf{J}\cdot\Delta\mathbf{s} \tag{15}$$

where

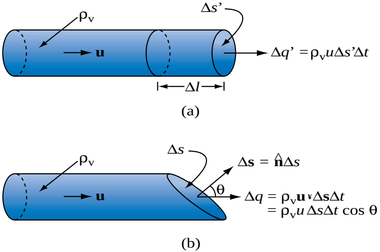


Figure 2: Charges with velocity $\mathbf{u}$ moving through a cross section $\Delta s'$ in (a) and Δs in (b).

$$\mathbf{J} = \rho_{\mathrm{v}} \mathbf{u} \quad (\mathrm{A/m^2}) \tag{16}$$

is the current density in $\mathrm{A/m^2}$. If $\mathbf{J}$ is known, the current is obtained by integration:

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \quad (\mathrm{A}) \tag{17}$$

There are different causes of current flow: namely, convection and conduction currents. When the charged particles are moving to cause the current it is a convection current. When the particles themselves do not move through the space it is a conduction current. For example, in a wire the same electrons do not move through the wire from one end to the other. For our problems, the conduction current will be more interesting and relevant.

4.3. Coulomb's Law

How do we relate the electric field density $\mathbf{E}$ (and corresponding electric flux density $\mathbf{D}$ with charge (possibly distributed)?

Statement of Coulomb's law:

1. An isolated charge q induces an electric field $\mathbf{E}$ such that

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \quad \text{(V/m)} \tag{18}$$

   where $\hat{\mathbf{R}}$ is a unit vector pointing from q to a point P (see Fig. 3). ε is electrical permittivity of the medium.

2. Given the presence of $\mathbf{E}$, the force acting on a test charge q' introduced at point P is

$$\mathbf{F} = q'\mathbf{E} \quad \text{(N)} \tag{19}$$

Note that force is measured in newtons (N), charge in coulombs (C) and $\mathbf{E}$ in N/C (= V/m).

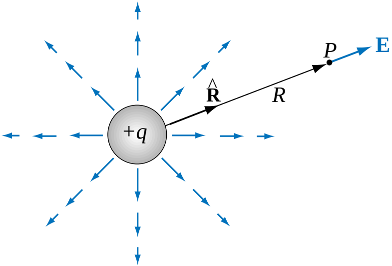


Figure 3: Electric-field lines due to a charge q.

The electric field intensity and flux density are related through

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \varepsilon = \varepsilon_r \varepsilon_0 \tag{20}$$

where $\varepsilon_0 = 8.854 \times 10^{-12}$ (F/m). ε_r is relative permittivity or dielectric constant.

Two important observations:

- If ε is independent of the magnitude of $\mathbf{E}$ then the material is linear (refers to the fact that $\mathbf{D}$ and $\mathbf{E}$ relate to each other linearly)
- If ε is independent of the direction of $\mathbf{E}$ $\Rightarrow$ the material is isotropic.

- **Electric field due to multiple point charges**

Let's have a look at what happens when two charges are present as in Fig. 4 and we are interested in the electric field at point P. We can apply superposition of the two electric fields. Start with q_1 charge at position $\mathbf{R}_1$

$$\mathbf{E}_1 = \frac{q_1(\mathbf{R} - \mathbf{R}_1)}{4\pi\varepsilon|\mathbf{R} - \mathbf{R}_1|^3} \quad \text{(V/m)} \tag{21}$$

followed by q_2 charge at position $\mathbf{R}_2$

$$\mathbf{E}_2 = \frac{q_2(\mathbf{R} - \mathbf{R}_2)}{4\pi\varepsilon|\mathbf{R} - \mathbf{R}_2|^3} \quad \text{(V/m)} \tag{22}$$

Where we have replaced the distance between the point charge q_1 and P with $R = |\mathbf{R} - \mathbf{R}_1|$ and the unit vector $\hat{\mathbf{R}} = \frac{\mathbf{R}-\mathbf{R}_1}{|\mathbf{R}-\mathbf{R}_1|}$.

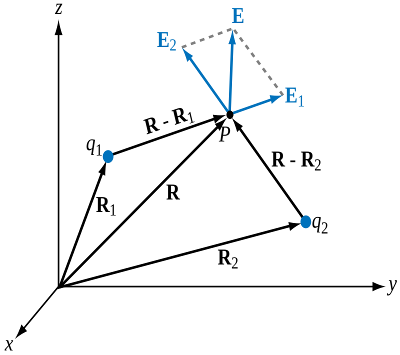


Figure 4: The electric field **E** at P due to two charges is equal to the vector sum of $\mathbf{E}_1$ and $\mathbf{E}_2$.

The geometry is clear on Fig. 4 but the math does not look nice:

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 \\ &= \frac{1}{4\pi\varepsilon}\left[\frac{q_1(\mathbf{R}-\mathbf{R}_1)}{|\mathbf{R}-\mathbf{R}_1|^3} + \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{|\mathbf{R}-\mathbf{R}_2|^3}\right]\end{aligned} \tag{23}$$

which can then be extended to any number of charges:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i(\mathbf{R}-\mathbf{R}_i)}{|\mathbf{R}-\mathbf{R}_i|^3} \quad (\mathrm{V/m}) \tag{24}$$

- **Electric field due to a charge distribution**

When we have a distribution of charges present it is a bit more complicated. We can use some calculus tricks: restrict the charge to a small volume $d\nu'$ so that the small (differential) charge is $dq = \rho_{\mathrm{v}} d\nu'$

$$d\mathbf{E} = \hat{\mathbf{R}}' \frac{dq}{4\pi\varepsilon R'^2} = \hat{\mathbf{R}}' \frac{\rho_{\mathrm{v}}\, dv'}{4\pi\varepsilon R'^2} \tag{25}$$

which is shown in Fig. 5

To get the total field ⇒ use superposition, i.e. add up the $d\mathbf{E}$, i.e. integrate

$$\mathbf{E} = \int_{v'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{v'} \hat{\mathbf{R}}' \frac{\rho_{\mathrm{v}}\, dv'}{R'^2} \quad \text{(volume distribution)} \tag{26}$$

Note that both R' and $\hat{\mathbf{R}}'$ are functions of position on the integration volume ν'.

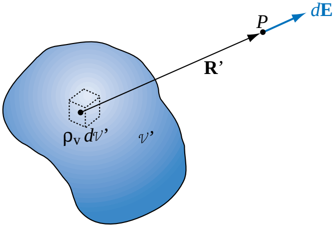


Figure 5: Electric field due to a volume charge distribution.

Extending this to surface and line charge distributions, we get:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2} \quad \text{(surface distribution)} \tag{27}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_{l'} \hat{\mathbf{R}}' \frac{\rho_l \, dl'}{R'^2} \quad \text{(line distribution)} \tag{28}$$

4.4. Gauss's Law

Gauss's law was given before:

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}} \quad \text{(Gauss's law)} \tag{29}$$

and this is called the *differential* form, to indicate spatial derivaties are used. There is also an integral form of Gauss's Law and is completely equivalent; we use the most convenient one. Here is how that comes about; take the volume integral of both sides of Gauss's Law in differential form,

$$\int_v \nabla \cdot \mathbf{D}\, dv = \int_v \rho_{\mathrm{v}}\, dv = Q \tag{30}$$

where Q is total charge enclosed by ν. Use divergence theorem (eq. 3-98)

$$\int_v \nabla \cdot \mathbf{D}\, dv = \oint_S \mathbf{D} \cdot d\mathbf{s} \tag{31}$$

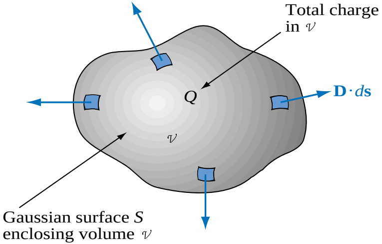


Figure 6: Gauss's law states that the outward flux of **D** through a surface is proportional to the enclosed charge Q.

which by comparison gives

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \quad \text{(Gauss's law)} \tag{32}$$

The integral form is illustrated in Fig. 6.

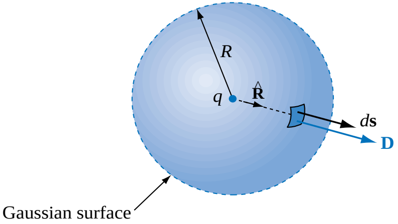


Figure 7: Electric field **D** due to point charge q.

If the electric flux density **D** is evaluated far away from some charge contained inside small volume $\Delta\nu$, then the charge can be treated as a **point charge**. We can use integral form of Gauss's law to calculate **D** for a point charge as illustrated in Fig. 7.

Symmetry dictates that magnitude of $\mathbf{D}$ must be the same at all points on the Gaussian surface S.

$$\Rightarrow \mathbf{D}(\mathbf{R}) = \hat{\mathbf{R}} D_R \tag{33}$$

(which coordinate system?). Similarly, $d\mathbf{s} = \hat{\mathbf{R}} ds$. Apply Gauss's law to this electric flux density

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \oint_S \hat{\mathbf{R}} D_R \cdot \hat{\mathbf{R}}\, ds = \oint_S D_R\, ds = \int_0^{2\pi} \int_0^{\pi} D_R R^2 \sin\theta d\theta d\phi \tag{34}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = D_R(4\pi R^2) = q \tag{35}$$

$$D_R = \frac{q}{4\pi R^2} \tag{36}$$

Plug this back into eq. 33, to get

$$\mathbf{E}(\mathbf{R}) = \frac{\mathbf{D}(\mathbf{R})}{\varepsilon} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \quad (\mathrm{V/m}) \tag{37}$$

which is the same as eq. 18, i.e. Coulomb's law.

What's the big deal? Gauss's law is easier to apply, provided that some *symmetry* of the charge distribution exists, so that we can "guess" what the variation of magnitude and direction of **D** is going to be. For example, use surfaces such that the magnitude of **D** is constant on the surface and the direction is either normal to the surface or tangential.

Find expression for **E** in free space due to an infinitely long line of charge with uniform charge density ρ_l along the z-axis.

Symmetry dictates that $\mathbf{D} = \hat{\mathbf{r}} D_r$. Find the right surface for integration — see Fig. 8.

$$\int_{z=0}^{h} \int_{\phi=0}^{2\pi} \hat{\mathbf{r}} D_r \cdot \hat{\mathbf{r}} r d\phi dz = \rho_l h \Rightarrow 2\pi h D_r r = \rho_l h \tag{38}$$

so that

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \frac{\rho_l}{2\pi\varepsilon_0 r} \quad \text{(infinite line of charge)} \tag{39}$$

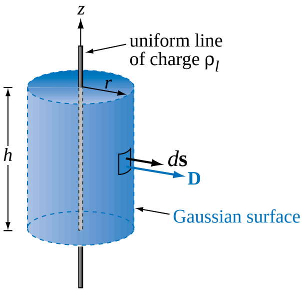


Figure 8: Gaussian surface around an infinitely long line of charge (Example 4-6).

4.5. Electric scalar potential

How are electric fields related to our circuit analysis? It's not immediately obvious, but the voltage we use in circuits actually represents changes in **potential energy**, i.e. energy required to move a unit charge between two points. The voltage difference represents the amount of potential energy (or work) needed to move a unit charge from one point to another. Let's examine this new quantity in detail and the relationship between electric scalar potential and electric field.

- **Electric potential as a function of electric field**

Take positive charge q in uniform field, fig. 9. The force on that charge will be pushing it in the direction of the field, i.e. $-y$. To make it move at a constant velocity, forces have to be balanced, i.e. an external force would have to "push" the charge in the $+y$ direction so that

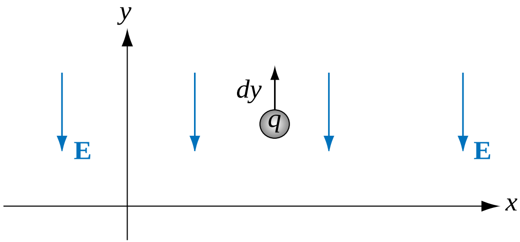


Figure 9: Work done in moving a charge q a distance dy against the electric field $\mathbf{E}$ is $dW = qE\,dy$.

$$\mathbf{F}_{ext} = -\mathbf{F}_e = -q\mathbf{E} \tag{40}$$

What is the work (energy) expanded?

$$dW = \mathbf{F}_{ext} \cdot d\mathbf{l} = -q\mathbf{E} \cdot d\mathbf{l} \quad \text{(J)} \tag{41}$$

and is measured in Joules (J). If charge is moved some distance dy along $\hat{\mathbf{y}}$, then the differential potential energy is,

$$dW = -q(-\hat{\mathbf{y}}E) \cdot \hat{\mathbf{y}}\, dy = qE\, dy \tag{42}$$

The differential electric potential (differential voltage dV) is the differential potential energy dW per unit charge,

$$dV = \frac{dW}{q} = -\mathbf{E} \cdot d\mathbf{l} \quad \text{(J/C or V)} \tag{43}$$

Since voltage is measured in volts ⇒ el. field measured in volts/meter

Given two points, potential difference between them is obtained by integration

$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l} \tag{44}$$

or

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l} \tag{45}$$

where V_1, V_2 are electric potentials at those points.

Integration path should not matter; why? Gravitational field analogy.

If we have a closed loop, then integration will yield zero! That should be expected from Kirchoff's voltage law. More generally,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \quad \text{(Electrostatics)} \tag{46}$$

any closed loop integration of electrostatic field $\mathbf{E}$ is zero $\Rightarrow$ this is conservative (irrotational) field.

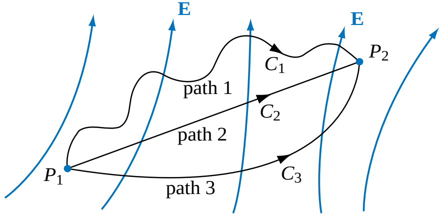


Figure 10: In electrostatics, the potential difference between P_2 and P_1 is the same irrespective of the path used for calculating the line integral of the electric field between them.

This property is also expressed in Maxwell's 2nd equation

$$\nabla \times \mathbf{E} = 0 \tag{47}$$

Take a surface integral of above and use Stokes's theorem to convert it to a line integral

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = \oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \tag{48}$$

where C is a closed contour surrounding S. Differential vs. integral form!

Going back to potential: remember that voltage in a circuit is meaningless unless there is a defined point of zero potential, called **ground** so that all voltages are expressed relative to that point. $\Rightarrow$ use the same principle to electric potential and define the reference point potential at infinity, so that $V_1 = 0$ when P_1 is at ∞, so that

$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \quad \text{(V)} \tag{49}$$

- **Electric potential due to point charges**

We've derived **E** for point charges:

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \quad \text{(V/m)} \tag{50}$$

how do we get potential?

$$V = -\int_{\infty}^{R} \left(\hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \right) \cdot \hat{\mathbf{R}}\, dR = \frac{q}{4\pi\varepsilon R} \quad \text{(V)} \tag{51}$$

where we've chosen the easiest path for integration ($d\hat{l} = \hat{R} dR$). this can be generalized to any other "origin"

$$V(\mathbf{R}) = \frac{q}{4\pi\varepsilon |\mathbf{R} - \mathbf{R}_1|} \quad \text{(V)} \tag{52}$$

Superposition still valid:

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{R} - \mathbf{R}_i|} \quad \text{(V)} \tag{53}$$

- **Electric potential due to continuous distributions**

Trick: instead of "true" point charge, look at some "small" volume, surface or line segment. Instead of summation, use integration. Also, redefine distance $R' = |\mathbf{R} - \mathbf{R}_i|$. $\Rightarrow$

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{\rho_{\mathrm{v}}}{R'} \, dv' \quad \text{(volume distribution)} \tag{54}$$

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{S'} \frac{\rho_s}{R'} \, ds' \quad \text{(surface distribution)} \tag{55}$$

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{l'} \frac{\rho_l}{R'} \, dl' \quad \text{(line distribution)} \tag{56}$$

- **Electric field as a function of electric potential**

Turn the tables and express $\mathbf{E}$ in terms of V. Start with

$$dV = -\mathbf{E} \cdot d\mathbf{l} \tag{57}$$

From before (eq. 3.73) we know that for scalar function

$$dV = \nabla V \cdot d\mathbf{l} \tag{58}$$

Comparing the two ⇒

$$\mathbf{E} = -\nabla V \tag{59}$$

So, what's the big deal? Using expressions developed above, we can calculate el. potential V; note that the eqs. 54 etc. have scalar integrals which are much easier to calculate. Once we get V, finding $\mathbf{E}$ is simply taking a gradient.

Do example 4-7. Electric field of an *electric dipole*. First look at as a simple sum of two contributions. Geometry of the case leads to these approximations:

$$R_2 - R_1 \approx d\cos\Theta, \quad R_1 R_2 \approx R^2 \tag{60}$$

so that

$$V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2} \tag{61}$$

Re-write the numerator as a dot product of $q\mathbf{d}$ (see fig. 11):

$$qd\cos\theta = q\mathbf{d}\cdot\hat{\mathbf{R}} = \mathbf{p}\cdot\hat{\mathbf{R}} \tag{62}$$

$\mathbf{p} = q\mathbf{d}$ is **dipole moment** of the electric dipole.

$$V = \frac{\mathbf{p}\cdot\hat{\mathbf{R}}}{4\pi\varepsilon_0 R^2} \quad \text{(electric dipole)} \tag{63}$$

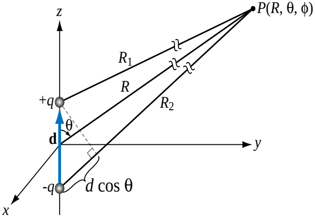
What is el. field? In spherical coordinates:

$$\mathbf{E} = -\nabla V = -\left(\hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\right) \tag{64}$$

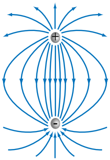
so that

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}\left(\hat{\mathbf{R}}2\cos\theta + \hat{\theta}\sin\theta\right) \quad \text{(V/m)} \tag{65}$$

Note that this is valid only for $R \gg d$. If not, el. field profile calculated from original formula (sum of two contributions). Result shown in fig. 11.



(a) Electric dipole



(b) Electric-field pattern

Figure 11: Electric dipole with dipole moment $\mathbf{p} = q\mathbf{d}$ (Example 4-7).

• Poisson's equation

There are other ways to write Gauss's law. Use $\mathbf{D} = \epsilon\mathbf{E}$ so that

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\mathrm{v}}}{\varepsilon} \tag{66}$$

but we can now express el. field as - gradient of potential

$$\nabla \cdot (\nabla V) = -\frac{\rho_{\mathrm{v}}}{\varepsilon} \tag{67}$$

and this is, by definition, the Laplacian of a scalar function

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{68}$$

so, in this notation

$$\nabla^2 V = -\frac{\rho_{\mathrm{v}}}{\varepsilon} \quad \text{(Poisson's equation)} \tag{69}$$

Equation for V we derived earlier

$$V = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{\rho_{\mathrm{v}}}{R'}\, dv' \tag{70}$$

satisfies Poisson's equation.

If medium contains no charges

$$\nabla^2 V = 0 \quad \text{(Laplace's equation)} \tag{71}$$

Use Poisson's and Laplace's eqs. for determining V in regions where boundary conditions are known (C, p-n junctions etc).

4.6. Conductors

From the electromagnetic perspective, we need three parameters, the constitutive parameters, to characterize a medium: electrical permittivity ε, magnetic permeability μ and conductivity σ. For homogeneous material the constitutive parameters are constant. For isotropic material the constitutive parameters do not depend on direction.

The conductivity measures how easily electrons travel through material under the influence of external field: conductors (metals) or dielectrics (insulators) and semiconductors.

Averaged electron movement is described by *electron drift velocity* $\mathbf{u}_e$ which gives rise to *conduction current*.

Perfect conductors have infinite conductivity while insulators have zero conductivity. Materials with conductivities between are categorized as semi-conductors. Most metals have large but finite conductivities, materials that have near infinite conductivity are called superconductors.

What's the relationship between velocity and the externally applied electric field?

$$\mathbf{u}_{\text{e}} = -\mu_{\text{e}}\mathbf{E} \quad (\text{m/s}) \tag{72}$$

where μ_{e} is electron mobility $(\text{m}^2/\text{V}-\text{s})$. If you had positive charges, you'd need to change the sign

$$\mathbf{u}_{\text{h}} = \mu_{\text{h}}\mathbf{E} \quad (\text{m/s}) \tag{73}$$

These are the holes that are positive charge carriers.

If we assume volume density of charge ρ_{v} then these charges produce current density $\mathbf{J} = \rho_{\text{v}}\mathbf{u}$. If both types of charges are present:

$$\mathbf{J} = \mathbf{J}_{\text{e}} + \mathbf{J}_{\text{h}} = \rho_{\text{ve}}\mathbf{u}_{\text{e}} + \rho_{\text{vh}}\mathbf{u}_{\text{h}} \quad (\text{A/m}^2) \tag{74}$$

which after substitution of eqs. 72, 73 gives

$$\mathbf{J} = (-\rho_{\text{ve}}\mu_{\text{e}} + \rho_{\text{vh}}\mu_{\text{h}})\mathbf{E} \tag{75}$$

The quantity in parenthesis is the conductivity (recall, $\mathbf{J} = \sigma\mathbf{E}$). If we take N_{e} and N_{h} to be the number of free electrons and holes per unit volume, the charge density can be written as, $\rho_{\mathrm{ve}} = -N_{\mathrm{e}}e$ and $\rho_{\mathrm{vh}} = N_{\mathrm{h}}e$. This give us,

$$\sigma = -\rho_{\mathrm{ve}}\mu_{\mathrm{e}} + \rho_{\mathrm{vh}}\mu_{\mathrm{h}} = (N_{\mathrm{e}}\mu_{\mathrm{e}} + N_{\mathrm{h}}\mu_{\mathrm{h}})e \quad \text{(S/m) (semiconductor)} \tag{76}$$

For metals only electrons contribute

$$\sigma = -\rho_{\mathrm{ve}}\mu_{\mathrm{e}} = N_{\mathrm{e}}\mu_{\mathrm{e}}e \quad \text{(S/m) (conductor)} \tag{77}$$

and both satisfy Ohm's Law:

$$\mathbf{J} = \sigma\mathbf{E} \quad (\text{A/m}^2) \text{ (Ohm's law)} \tag{78}$$

For a perfect dielectric: $\sigma = 0, \mathbf{J} = 0$ regardless of $\mathbf{E}$. In a perfect conductor $\sigma = \infty$, $\mathbf{E} = \mathbf{J}/\sigma = 0$ regardless of $\mathbf{J}$.

$$\begin{aligned} \text{Perfect dielectric:} \quad & \mathbf{J} = 0 \\ \text{Perfect conductor:} \quad & \mathbf{E} = 0 \end{aligned}$$

Good metals are very close to perfect conductors. If $\mathbf{E} = 0 \Rightarrow$ no change in electric potential! We call this medium *equipotential*.

This also follows from the definition of the voltage difference between two points is the line integral of the electric file between the points. Since, in a conductor, the electric field is zero everywhere in the conductor then the voltage difference is also zero.

• Resistance

We can apply $\mathbf{J} = \sigma \mathbf{E}$ to obtain resistance. The setup given in Fig. 12. The voltage applied between points 1 and 2 establishes electric field $\mathbf{E} = \hat{\mathbf{x}} E_x$ and points from the higher potential (1) to the lower potential (2).

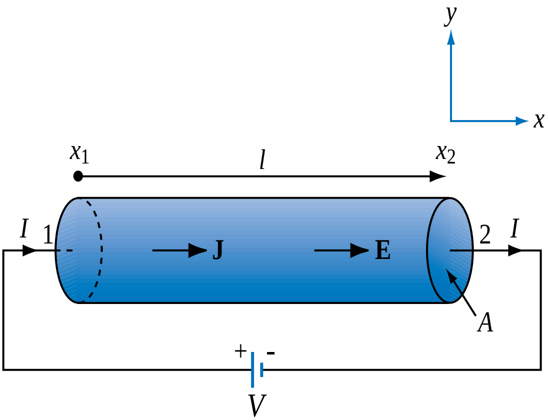


Figure 12: Linear resistor of cross section A and length l connected to a d-c voltage source V.

The potential difference can be written,

$$V = V_1 - V_2 = -\int_{x_2}^{x_1} \mathbf{E} \cdot d\mathbf{l} = -\int_{x_2}^{x_1} \hat{\mathbf{x}} E_x \cdot \hat{\mathbf{x}}\, dl = E_x l \quad \text{(V)} \tag{79}$$

What is the current flowing through intersection A?

$$I = \int_A \mathbf{J} \cdot d\mathbf{s} = \int_A \sigma \mathbf{E} \cdot d\mathbf{s} = \sigma E_x A \quad \text{(A)} \tag{80}$$

To get resistance ($R = V/I$) take a ratio of eqs. 79 and 80:

$$R = \frac{l}{\sigma A} \quad (\Omega) \tag{81}$$

This can be generalized to arbitrary shape,

$$R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} \tag{82}$$

To calculate conductance (1/R) we use

$$G = \frac{1}{R} = \frac{\sigma A}{l} \quad \text{(S)} \tag{83}$$

for the linear resistor. For the coaxial cable the conductance is as follows:

$$\mathbf{J} = \hat{\mathbf{r}}\frac{I}{A} = \hat{\mathbf{r}}\frac{I}{2\pi r l} \tag{84}$$

$$\mathbf{E} = \hat{\mathbf{r}}\frac{I}{2\pi\sigma r l} \tag{85}$$

$$V_{ab} = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = -\int_b^a \frac{I}{2\pi\sigma l}\frac{\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\,dr}{r} = \frac{I}{2\pi\sigma l}\ln\left(\frac{b}{a}\right) \tag{86}$$

$$G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{ab} l} = \frac{2\pi\sigma}{\ln(b/a)} \quad \text{(S/m)} \tag{87}$$

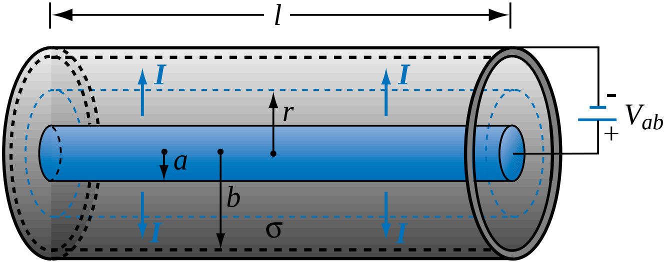


Figure 13: Coaxial cable of Example 4-9.

• Joule's Law

What's the power dissipated in a conducting medium when $\mathbf{E}$ is present? Look at the case when both positive (holes) and negative (electron) charges are present with volume charge densities $\rho_{\mathrm{vh}}, \rho_{\mathrm{ve}}$. The force on q_{e} and q_{h} are $\mathbf{F}_{\mathrm{e}} = q_{\mathrm{e}}\mathbf{E} = \rho_{\mathrm{ve}}\mathbf{E}\Delta v$ and $\mathbf{F}_{\mathrm{h}} = q_{\mathrm{h}}\mathbf{E} = \rho_{\mathrm{vh}}\mathbf{E}\Delta v$

The differential distance to move a charge is Δl_{e} and Δl_{h} and the work (energy) is force times distance, i.e.

$$\Delta W = \mathbf{F}_{\mathrm{e}} \cdot \Delta\mathbf{l}_{\mathrm{e}} + \mathbf{F}_{\mathrm{h}} \cdot \Delta\mathbf{l}_{\mathrm{h}} \tag{88}$$

Power P is time rate of change of energy, measured in watts.

$$\begin{aligned} \Delta P &= \frac{\Delta W}{\Delta t} = \mathbf{F}_{\mathrm{e}} \cdot \frac{\Delta\mathbf{l}_{\mathrm{e}}}{\Delta t} + \mathbf{F}_{\mathrm{h}} \cdot \frac{\Delta\mathbf{l}_{\mathrm{h}}}{\Delta t} = \mathbf{F}_{\mathrm{e}} \cdot \mathbf{u}_{\mathrm{e}} + \mathbf{F}_{\mathrm{h}} \cdot \mathbf{u}_{\mathrm{h}} \\ &= (\rho_{\mathrm{ve}}\mathbf{E} \cdot \mathbf{u}_{\mathrm{e}} + \rho_{\mathrm{vh}}\mathbf{E} \cdot \mathbf{u}_{\mathrm{h}})\Delta v = \mathbf{E} \cdot \mathbf{J}\Delta v \end{aligned} \tag{89}$$

were we use $\mathbf{J} = \rho_{\mathrm{v}}\mathbf{u}$.

Finally, the total dissipated power in a volume ν is

$$P = \int_v \mathbf{E} \cdot \mathbf{J} dv \quad \text{(W) (Joule's law)} \tag{90}$$

which is Joule's law; using $\mathbf{J} = \sigma \mathbf{E}$ results in

$$P = \int_v \sigma |\mathbf{E}|^2 \, dv \quad \text{(W)} \tag{91}$$

This can be simplified further for the uniform resistor case above by breaking the volume integral into surface and line integral.

$$P = \int_v \sigma |\mathbf{E}|^2 \, dv = \int_A \sigma E_x \, ds \;\int_l E_x \, dl = (\sigma E_x A)(E_x l) = IV \quad \text{(W)} \tag{92}$$

(using previous results for V and I). Using $V = IR$,

$$P = I^2 R \quad \text{(W)} \tag{93}$$

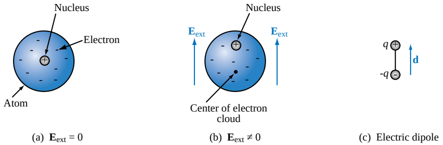


Figure 14: In the absence of an external electric field $\mathbf{E}_{\text{ext}}$, the center of the electron cloud is co-located with the center of the nucleus, but when a field is applied, the two centers are separated by a distance d.

4.7. Dielectrics

The difference between dielectrics (insulators) and conductors is the availability of free electrons in conductors. These can move in conductors when an external field is applied. Figure 14 illustrates what happens in dialectrics.

- Since dielectrics are insulators, an applied external electric field cannot induce charge movement similar to what happens in conductors.

- There is a change of "balance" between positive and negative charges in atoms (or molecules). A distortion of the atom can occur which acts to polarize the material.

- Effectively, the external electric field sets up a dipole, illustrated in Fig. 14.

- This *induced* or *polarization* field is weaker and opposite in direction to $\mathbf{E}_{ext} \Rightarrow$ the net electric field in dielectrics is smaller than $\mathbf{E}_{ext}$.

- These dipoles align themselves as in Fig. 15. Note that there is positive charge on top and negative on the bottom surface (for electric field from bottom to top)

- Things are a bit different for *polar* materials, i.e. those that already have dipoles even when no external field is present. Such dipoles would tend to align along the lines of external electric field, similarly to fig. 15.

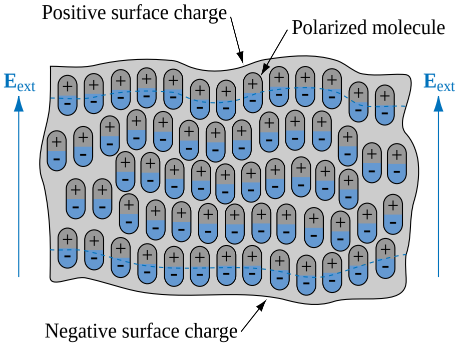


Figure 15: A dielectric medium polarized by an external electric field $\mathbf{E}_{\text{ext}}$.

How is this going to affect the relationship between the electric field intensity **E** and electric flux density **D**?

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{94}$$

where **P** is the electric polarization field, and accounts for the polarization properties of the materials. It is produced by the electric field **E** and depends on material properties. Materials classified as:

Linear: Magnitude of the induced polarization field is directly proportional to the magnitude of **E**

Isotropic: Polarization field **P** and **E** are in the same direction

Anisotropic: **P** and **E** have different directions

Homogeneous: Material properties (constitutive parameters), i.e. ε, μ, σ are constant throughout medium

For linear, isotropic and homogenous media:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \tag{95}$$

where χ_e is the electric susceptibility of the material.

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E} \tag{96}$$

so that permittivity ε is

$$\varepsilon = \varepsilon_0 (1 + \chi_e) \tag{97}$$

Often, it is more convenient to use relative permittivity $\varepsilon_r = \varepsilon / \varepsilon_0$. See table 4-2 for the relative permittivities for various materials.

- For most conductors $\varepsilon_r \approx 1$
- Air has dielectric constant ≈ 1
- If $\mathbf{E}$ exceeds a critical value, called the dielectric strength, electrons are stripped away and dielectric starts conducting $\Rightarrow$ dielectric breakdown.

4.8. Electric boundary conditions

We examined only simple situations where we are in a single medium that is not changing. If there is a change from one medium to another, we need to examine what happens to the electric field at the interface. In other words, we need to know the boundary conditions (BC-s). We'll look at both dielectric-dielectric and dielectric-conductor boundaries. It's important to note that these BC-s will be valid for time-dependent cases as well.

- Start with Fig. 16 that illustrates the boundary between two dielectrics.

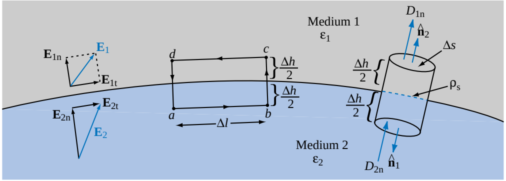


Figure 16: Interface between two dielectric media.

- In general, the boundary may contain surface charge density ρ_s.
- We need to examine both the tangential and normal components of the electric field. We start with the tangential by constructing a closed rectangular loop *abcda* in Fig. 16.
- Apply the conservative property of the electric field $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$ which states that line integral over a closed loop produces zero.

- Also let $\Delta h \to 0$ so that bc and da contributions $\to 0$.

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{E}_2 \cdot d\mathbf{l} + \int_c^d \mathbf{E}_1 \cdot d\mathbf{l} = 0 \tag{98}$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are the electric field in media 1 and 2.

- Breaking up the field into normal and tangential components gives,

$$\mathbf{E}_1 = \mathbf{E}_{1t} + \mathbf{E}_{1n}, \quad \mathbf{E}_2 = \mathbf{E}_{2t} + \mathbf{E}_{2n} \tag{99}$$

- Over segment ab, $\mathbf{E}_t \cdot \Delta l$ has the opposite sign of $\mathbf{E}_t \cdot \Delta l$ over cd (why?) so that,

$$E_{2t}\,\Delta l = E_{1t}\,\Delta l \quad \text{or} \quad E_{1t} = E_{2t} \quad \text{(V/m)} \tag{100}$$

- **Tangential component of the electric field is continuous across the boundary between any two media**.

- For the flux density we use $D_{1t} = \varepsilon_1 \mathbf{E}_{1t}$ and $D_{2t} = \varepsilon_2 \mathbf{E}_{2t}$

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2} \tag{101}$$

- What about the normal components? We use Gauss's law.

- First, set up a small cylinder, as shown in Fig. 16. Gauss's law states that the total **outward** flux of $\mathbf{D}$ integrated over the three surfaces must be equal to the total charge inside the cylinder.

- As we let $\Delta h \to 0$, the side (curved) surface contribution $\to 0$. This also means that any volume charge density will not contribute to the total charge (i.e. only have to consider surface charge).

- The only remaining charge is the one at the boundary: $Q = \rho_s \Delta s$. Add up the flux from the other boundaries:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{\text{top}} \mathbf{D}_1 \cdot \hat{\mathbf{n}}_2 \, ds + \int_{\text{bottom}} \mathbf{D}_2 \cdot \hat{\mathbf{n}}_1 \, ds = \rho_s \, \Delta s \quad (102)$$

  where $\hat{\mathbf{n}}$-s are outward normal unit vectors of the bottom and top surfaces.

- Since $\hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2$

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad (\mathrm{C/m^2}) \tag{103}$$

- By looking at the normal components, which are defined as being along the normal unit vectors, we get

$$D_{1n} - D_{2n} = \rho_s \quad (\mathrm{C/m^2}) \tag{104}$$

  i.e. **normal component of D changes abruptly at a charged boundary between two different media, and the amount of change is equal to the surface charge density.**

- What about the electric field?

$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s \tag{105}$$

Summary: conservative property of $\mathbf{E}$

$$\nabla \times \mathbf{E} = 0 \Longleftrightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \tag{106}$$

leads to continuous tangential components of **E** across a boundary, while the divergence property of **D**

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}} \Longleftrightarrow \oint_S \mathbf{D} \cdot d\mathbf{s} = Q \tag{107}$$

leads to abrupt changes in normal components of **D** across the boundary. See Table 4-3 for a summary of BC-s.

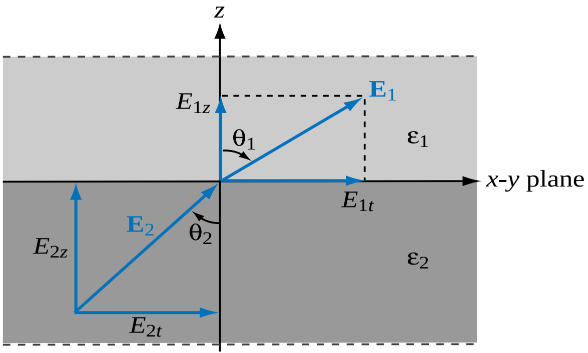


Figure 17: BC-s between two dielectric media (Ex. 4-10).

- **Dielectric-conductor boundary**

Medium 2 is now a perfect conductor, i.e. $\mathbf{E} = \mathbf{D} = 0$ everywhere inside medium 2.

- So, $E_{2t} = 0$, $D_{2n} = 0$ and previously we found that $E_{1t} = E_{2t}$ and $D_{1n} - D_{2n} = \rho_s$ so

$$E_{1t} = D_{1t} = 0, \quad \text{and} \quad D_{1n} = \varepsilon_1 E_{1n} = \rho_s \tag{108}$$

- Combining these two

$$\mathbf{D}_1 = \varepsilon_1 \mathbf{E}_1 = \hat{\mathbf{n}} \rho_s \quad \text{(at conductor surface)} \tag{109}$$

where $\hat{n}$ is unit vector directed normally outward from the conducting surface.

- **Electric field lines point directly away from the conductor surface when ρ_s is positive and directly toward the conductor surface when ρ_s is negative.**

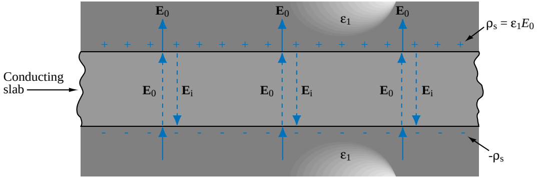


Figure 18: Conducting slab in an external electric field $\mathbf{E}_0$. Charges on the conductor surfaces induce an internal electric field $\mathbf{E}_i = -\mathbf{E}_0$.

- Fig. 18 shows infinitely long conducting slab in uniform externally applied $\mathbf{E}_0$. Since $\mathbf{E}_0$ points away from the upper surface it induces a positive charge density ρ_s. On the bottom surface it is opposite and we get $-\rho_s$. These surface charges induce an electric field $\mathbf{E}_\mathrm{i}$ in the conductor such that the total field in the conductor is zero (as it should be).

- The presence of surface charges can be viewed as inducing an electric field inside the conductor $\mathbf{E}_{\mathrm{i}}$, resulting in the total field $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_i$.
- In a perfect conductor $\mathbf{E} = 0 \Rightarrow \mathbf{E}_i = -\mathbf{E}_0$.

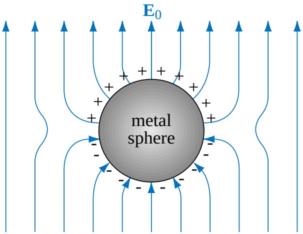


Figure 19: Metal sphere in an external electric field $\mathbf{E}_0$.

- Now take a look at what happens when a metal sphere is introduced in electric field, as in Fig. 19.
- Negative charge on the bottom, positive on top of sphere. electric field *bends* to satisfy condition $\mathbf{D}_1 = \varepsilon_1 \mathbf{E}_1 = \hat{\mathbf{n}}\rho_s$.
- **E is always normal to the surface at the conductor boundary!**

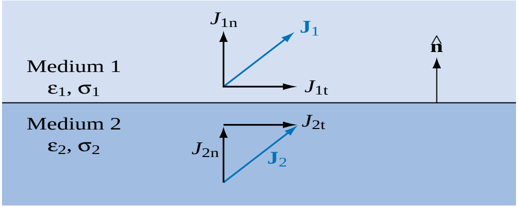


Figure 20: Boundary between two conducting media.

- **Conductor-conductor boundary**

Here we examine the boundary between materials that are not perfect conductors or perfect dielectrics as shown in Fig. 20. Note that we now also have conductivities σ.

- For electric field (and fluxes) things stay the same, so we use:

$$E_{1t} = E_{2t}, \quad \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s \tag{110}$$

- The two media have finite conductivities, so the electric fields give rise to current densities, i.e. $\mathbf{J}_1 = \sigma \mathbf{E}_1$ etc. so that eq. 110 leads to

$$\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}, \quad \varepsilon_1 \frac{J_{1n}}{\sigma_1} - \varepsilon_2 \frac{J_{2n}}{\sigma_2} = \rho_s \tag{111}$$

- The tangential components represent currents flowing parallel to the boundary ⇒ no charge transfer is involved between these two components.

- Normal components of current density are different: if $J_{1n} \neq J_{2n}$ then the amount of charge (per second) that arrives at the boundary is different than the amount that leaves ⇒ ρ_s would have to change with time! This cannot be allowed according to the "static" assumption that fields and charges are not time dependent.

- **Therefore, the normal component of J has to be continuous across the boundary between the two different media under electrostatic conditions**, i.e. set $J_{1n} = J_{2n}$

$$J_{1n} \left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2} \right) = \rho_s \quad \text{(electrostatics)} \tag{112}$$

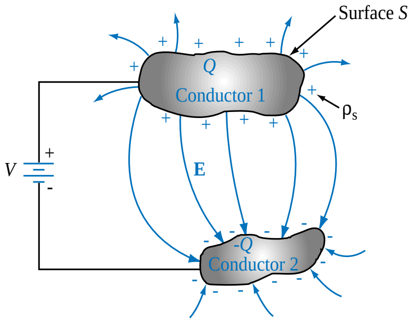


Figure 21: A d-c source connected to two conducting bodies (C).

4.9. Capacitance

A capacitor is formed whenever there are two metal (conducting) bodies separated by a dielectric.

- A d-c voltage applied to such pair, as shown in Fig. 21, will charge up the conductor that is connected to the + side of the source with $+Q$, and the other conductor with $-Q$.

- Note that when conductor has excess charge **it distributes the charge on its surface in such a manner as to maintain a zero electric field everywhere within the conductor.** This ensures that the conductor is an equipotential body, i.e. that the potential is the same everywhere in or on the conductor.

- Capacitance of a two-conductor capacitor is defined as

$$C = \frac{Q}{V} \quad \text{(C/V or F)} \tag{113}$$

- The charge on the surface gives rise to the electric field $\mathbf{E}$. Lines originate on + charges and terminate on − charges. Remember that $\mathbf{E}$ is always normal to conductor surface.

$$E_n = \hat{\mathbf{n}} \cdot \mathbf{E} = \frac{\rho_s}{\varepsilon} \quad \text{(at conductor surface)} \tag{114}$$

- What is Q equal to? We need to integrate over the surface:

$$Q = \int_S \rho_s \, ds = \int_S \varepsilon \hat{\mathbf{n}} \cdot \mathbf{E} \, ds = \int_S \varepsilon \mathbf{E} \cdot d\mathbf{s} \tag{115}$$

- Remember how to calculate voltage difference:

$$V = V_{12} = -\int_{P_2}^{P_1} \mathbf{E} \cdot d\mathbf{l} \tag{116}$$

  where P_1 is on conductor 1 and P_2 on conductor 2.

- Plug these into definition of C

$$C = \frac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}} \quad \text{(F)} \tag{117}$$

  where l is integration path from conductor 2 to conductor 1.

- Note that the value of C is independent of E, but it depends on geometry and permittivity (dielectric const.) of the insulating material.

If the dielectric is not perfect, there is some finite resistance, which is calculated using the general expression for resistance (eq. 4.71 in book)

$$R = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} \quad (\Omega) \tag{118}$$

and if medium is homogeneous (uniform σ, ε), then multiplying the above with,

$$C = \frac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}} \quad (\mathrm{F}) \tag{119}$$

$$RC = \frac{\varepsilon}{\sigma} \tag{120}$$

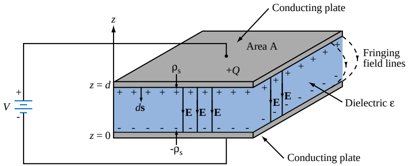


Figure 22: A d-c voltage source connected to a parallel-plate capacitor (Example 4-11).

Example 4-11: Capacitance and breakdown voltage of parallel plate capacitor. One approximation needed: neglect fringing fields! Setup shown in fig. 22.

$$V = -\int_0^d \mathbf{E} \cdot d\mathbf{l} = -\int_0^d (-\hat{\mathbf{z}}E) \cdot \hat{\mathbf{z}}\, dz = Ed \tag{121}$$

$$\rho_s = Q/A \tag{122}$$

$$\rho_s = Q/A \tag{123}$$

$$E = \rho_s/\epsilon = Q/\epsilon A \tag{124}$$

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d} \tag{125}$$

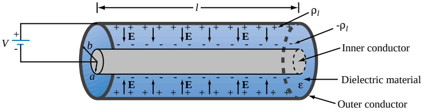


Figure 23: Coaxial capacitor filled with insulating material of permittivity ε (Example 4-12).

Example 4-12: capacitance of coaxial line.

$$\mathbf{E} = -\hat{\mathbf{r}}\frac{\rho_l}{2\pi\varepsilon r} = -\hat{\mathbf{r}}\frac{Q}{2\pi\varepsilon r l} \tag{126}$$

$$\begin{aligned} V = -\int_a^b \mathbf{E}\cdot d\mathbf{l} &= -\int_a^b \left(-\hat{\mathbf{r}}\frac{Q}{2\pi\varepsilon r l}\right)\cdot(\hat{\mathbf{r}}\,dr) \\ &= \frac{Q}{2\pi\varepsilon l}\ln\left(\frac{b}{a}\right) \end{aligned} \tag{127}$$

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon l}{\ln(b/a)} \tag{128}$$

$$C' = \frac{C}{l} = \frac{2\pi\varepsilon}{\ln(b/a)} \quad \text{(F/m)} \tag{129}$$

4.10. Electrostatic potential energy

Intuition tells us that placing charge onto capacitor plates will involve some energy expenditure. Where is it "spent"? If materials are perfect, then there are no ohmic losses. Energy is actually stored in the dielectric medium as electrostatic potential energy W_e and the amount is related to V, C and Q.

- Under the influence of an electric field, equal but opposite charges accumulate on the conductors. We can view it as a "transfer" of charge q from one conductor to another.

- From before, voltage accross the capacitor and q are related via

$$v = \frac{q}{C} \tag{130}$$

- By looking at eq. 43 (really a definition of electrostatic potential), the amount of work dW_{e} required to transfer additional charge dq is

$$dW_{\mathrm{e}} = v\,dq = \frac{q}{C}\,dq \tag{131}$$

- To get the total energy starting with an uncharged capacitor C we need to integrate:

$$W_{\mathrm{e}} = \int_0^Q \frac{q}{C}\,dq = \frac{1}{2}\frac{Q^2}{C} \quad \text{(J)} \tag{132}$$

or, (remembering that $Q = CV$)

$$W_{\mathrm{e}} = \frac{1}{2}\,CV^2 \quad \text{(J)} \tag{133}$$

- This can be expressed differently for parallel plate capacitor where, $C = \varepsilon A/d$, and $V = Ed$

$$W_{\mathrm{e}} = \frac{1}{2}\,\frac{\varepsilon A}{d}(Ed)^2 = \frac{1}{2}\,\varepsilon E^2(Ad) = \frac{1}{2}\,\varepsilon E^2 v \tag{134}$$

where $v = Ad$ is volume between the plates.

- This enables us to introduce *electrostatic energy density* w_e as W_e per unit volume

$$w_\mathrm{e} = \frac{W_\mathrm{e}}{v} = \frac{1}{2}\,\varepsilon E^2 \quad (\mathrm{J/m^3}) \tag{135}$$

although derived for a parallel plate capacitor it is equally valid valid in the general case of a dielectric medium in an electric field. **E**.

- If energy density is known (i.e. **E**), finding out total electrostatic potential energy involves volume integration

$$W_\mathrm{e} = \frac{1}{2}\int_v \varepsilon E^2\, dv \quad (\mathrm{J}) \tag{136}$$

A different interpretation is possible by looking at the forces on the two charged plates.

- If two plates are allowed to move closer by some differential distance $d\mathbf{l}$ under the force $\mathbf{F}$ while maintaining constant charge, the work done is

$$dW = \mathbf{F} \cdot d\mathbf{l} \tag{137}$$

- This energy has to come from somewhere; some of the potential energy stored in the dielectric is expended, such that,

$$dW = -dW_{\mathrm{e}} \tag{138}$$

- From before we know how to calculate a directional derivative using the gradient,

$$dW_{\mathrm{e}} = \nabla W_{\mathrm{e}} \cdot d\mathbf{l} \tag{139}$$

- Comparing this with $dW = \mathbf{F} \cdot d\mathbf{l}$ gives us,

$$\mathbf{F} = -\nabla W_e \quad (\mathrm{N}) \tag{140}$$

Note the assumption that the charges in the system are constant.

- This can be applied to a parallel plate capacitor starting from (where $C = \epsilon A/d$ and we will replace fixed distance d with the variable z),

$$W_{\mathrm{e}} = \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2 z}{2\varepsilon A} \tag{141}$$

- Which, after the gradient operation, gives

$$\mathbf{F} = -\nabla W_e = -\hat{\mathbf{z}}\frac{\partial}{\partial z}\left(\frac{Q^2 z}{2\varepsilon A}\right) = -\hat{\mathbf{z}}\left(\frac{Q^2}{2\varepsilon A}\right) \tag{142}$$

or, using $E = \rho_s/\epsilon = Q/\epsilon A$ or $Q = \varepsilon A E$

$$\mathbf{F} = -\hat{\mathbf{z}}\frac{\varepsilon A E^2}{2} \quad \text{(parallel-plate capacitor)} \tag{143}$$