Contents

3. Vector Analysis

We had it easy so far: all our quantities were **scalar** (remember that some of the quantities were complex, though). From now on, we need vectors which will describe dependence of various quantities (primarily electric **E** and magnetic **H** fields) in 3-D space. What follows is a review of vector algebra, coordinate systems and vector calculus.

Remember, a vector specifies both magnitude and direction of a quantity. For example, temperature is a scalar (number only) while velocity is a vector (speed and direction).

3.1. Vector Algebra

A vector is specified by its **magnitude** $A = |\mathbf{A}|$, and its direction which can be specified using a **unit vector** $\hat{\mathbf{a}}$, as illustrated in Fig. 1

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{a}}A \tag{1}$$

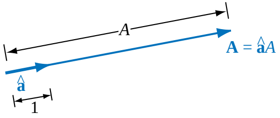


Figure 1: Vector $\mathbf{A} = \hat{\mathbf{a}}A$ has a magnitude $A = |\mathbf{A}|$ and unit vector $\hat{\mathbf{a}} = \mathbf{A}/A$.

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A} \tag{2}$$

In the Cartesian (or rectangular) coordinate system (shown in fig. 2), there are three mutually perpendicular coordinates x, y, z and corresponding unit (or *base*) vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. Any vector $\mathbf{A}$ can be represented in terms of its components along different axes, as illustrated in fig. 2.

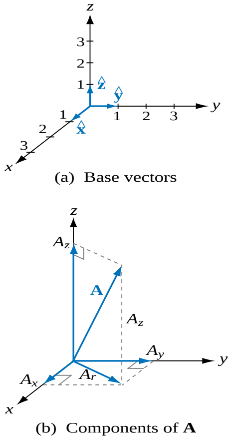


Figure 2: Cartesian coordinate system: (a) base vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, and (b) components of vector $\mathbf{A}$.

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z \tag{3}$$

Application of Pythagorean theorem, gives

$$A = |\mathbf{A}| = \sqrt[+]{A_x^2 + A_y^2 + A_z^2}, \quad \hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\sqrt[+]{A_x^2 + A_y^2 + A_z^2}} \tag{4}$$

You will also see vectors denoted $\hat{\mathbf{a}} = (A_x, A_y, A_z)$.

- **Vector equality**

Given two vectors

$$\mathbf{A} = \hat{\mathbf{a}}A = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z \quad (5)$$

$$\mathbf{B} = \hat{\mathbf{b}}B = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z \quad (6)$$

they will be equal, i.e. $\mathbf{A} = \mathbf{B}$, if they have equal magnitudes and identical unit vectors, i.e. $A = B$ and $\hat{\mathbf{a}} = \hat{\mathbf{b}}$, or $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$. It's interesting that two vectors can be equal but not identical. That is, they might have the same magnitude and direction but displaced from each other (think about parallel vectors).

- **Vector addition and subtraction**

The sum of two vectors is given by,

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{7}$$

where the order of addition does not matter. Graphical interpretation: parallelogram rule or head-to-tail rule (Fig. 3).

For rectangular coordinate system we have

$$\begin{aligned} \mathbf{C} &= \mathbf{A} + \mathbf{B} \\ &= (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) + (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) \\ &= \hat{\mathbf{x}}(A_x + B_x) + \hat{\mathbf{y}}(A_y + B_y) + \hat{\mathbf{z}}(A_z + B_z) \end{aligned} \tag{8}$$

i.e. summation is done by components.

Subtraction is done the same way as addition, but the negative vector's direction is changed,

$$\begin{aligned} \mathbf{D} = \mathbf{A} - \mathbf{B} &= \mathbf{A} + (-\mathbf{B}) \\ &= \hat{\mathbf{x}}(A_x - B_x) + \hat{\mathbf{y}}(A_y - B_y) + \hat{\mathbf{z}}(A_z - B_z) \end{aligned} \tag{9}$$

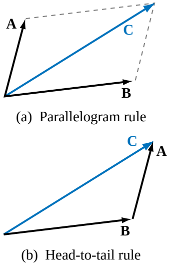


Figure 3: Vector addition by (a) the parallelogram rule and (b) the head-to-tail rule.

• Position and distance vectors

A position vector of the point P is defined as a vector starting from the origin and ending at P. Fig. 4 shows two position vectors, $\mathbf{R}_1$ and $\mathbf{R}_2$.

$$\mathbf{R}_1 = \overrightarrow{OP_1} = \hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1 \tag{10}$$

$$\mathbf{R}_2 = \overrightarrow{OP_2} = \hat{\mathbf{x}}x_2 + \hat{\mathbf{y}}y_2 + \hat{\mathbf{z}}z_2 \tag{11}$$

The vector connecting P_1 and P_2 is called the distance vector:

$$\begin{aligned} \mathbf{R}_{12} &= \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1 \\ &= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1) \end{aligned} \tag{12}$$

To find the distance between two points, calculate the magnitude of: $\mathbf{R}_{12}$

$$\begin{aligned} d &= |\mathbf{R}_{12}| \\ &= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} \end{aligned} \tag{13}$$

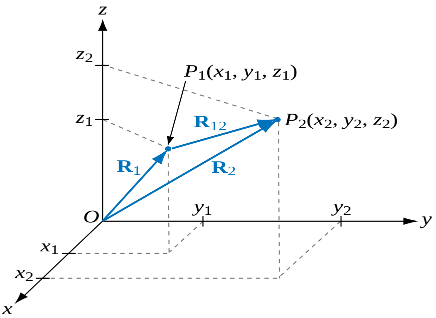


Figure 4: Position vector $\mathbf{R}_{12} = \overline{P_1\ P_2} = \mathbf{R}_2 - \mathbf{R}_1$.

The first subscript of $\mathbf{R}_{12}$ denotes the location of its tail and the second subscript the location of its head (see Fig. 4).

• Vector multiplication

There are three kinds of vector products:

- Simple product is a product between a scalar and a vector

$$\mathbf{B} = k\mathbf{A} = \hat{\mathbf{a}}kA = \hat{\mathbf{x}}(kA_x) + \hat{\mathbf{y}}(kA_y) + \hat{\mathbf{z}}(kA_z) \tag{14}$$

  This multiplication preserves direction, but changes the magnitude

- Scalar (dot) product is denoted by $\mathbf{A} \cdot \mathbf{B}$ and is defined as

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB} \tag{15}$$

  where θ_{AB} is the angle between vectors $\mathbf{A}$ and $\mathbf{B}$, as shown in Fig. 5.

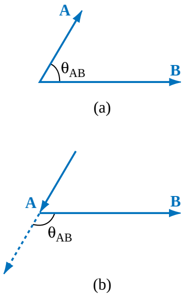


Figure 5: The angle θ_{AB} is the angle between **A** and **B** measured from **A** to **B** between vector tails. The dot product is positive if $0 \leq \theta_{AB} < 90°$, as in (A), and it is negative if $90° < \theta_{AB} \leq 180°$, as in (b).

Interpretation: $A\cos\Theta_{AB}$ is the projection of vector $\mathbf{A}$ along the direction of vector $\mathbf{B}$. Given,

$$\mathbf{A}\cdot\mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z)\cdot(\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) \tag{16}$$

and

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1 \tag{17}$$

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0 \tag{18}$$

we get

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z \tag{19}$$

Some properties of the dot product are:

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \quad \text{(commutative property)} \tag{20}$$

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \quad \text{(distributive property)} \tag{21}$$

Also, the dot product of a vector with itself gives,

$$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2 = A^2 \tag{22}$$

and

$$A = |\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}} \tag{23}$$

The angle between vectors can be determined from,

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{\sqrt[+]{\mathbf{A} \cdot \mathbf{A}}\,\sqrt[+]{\mathbf{B} \cdot \mathbf{B}}}\right] \tag{24}$$

- Vector or cross product is defined as

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin\theta_{AB} \tag{25}$$

wher θ_{AB} is the angle between **A** and **B** is measured from the tail of **A** to the tail of **B** (direction is important!).

Interpretation: cross product is equal (in magnitude) to the area of a parallelogram defined by two vectors and its direction is given by the right-hand rule (Fig. 6).

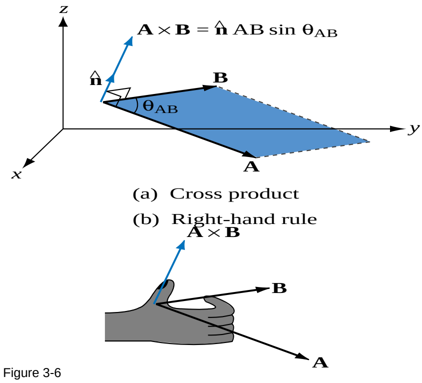


Figure 6: Cross product $\mathbf{A} \times \mathbf{B}$ points in the direction $\hat{\mathbf{n}}$, which is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$ and defined by the right-hand rule.

Some properties:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad \text{(anticommutative)} \tag{26}$$

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \quad \text{(distributive)} && (27) \\ \mathbf{A} \times \mathbf{A} &= 0 && (28) \end{aligned}$$

From the definition, we observe that

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \tag{29}$$

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0 \tag{30}$$

If we write out the product

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \times (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) \\ &= \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) \\ &\quad + \hat{\mathbf{z}}(A_xB_y - A_yB_x) \end{aligned} \tag{31}$$

but it is simpler to remember this

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{32}$$

• Triple products

Not all combinations of vector products are meaningful, e.g. $\mathbf{A} \times (\mathbf{B} \cdot \mathbf{C})$. What about $\mathbf{A}(\mathbf{B} \cdot \mathbf{C})$?

- Scalar triple product is a dot product of vector with a cross product of two other vectors, and it obeys cyclic order

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{33}$$

The result can be written in the form of a determinant

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{34}$$

- Vector triple product involves cross product of a vector with a cross product of two other vectors:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \tag{35}$$

It does not obey associative law, i.e.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \tag{36}$$

i.e. order of multiplication must be specified with parenthesis. Furthermore,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{37}$$

3.2. Orthogonal coordinate systems

Why do we care about coordinate systems other than the familiar Cartesian? Solving specific problems can be simplified greatly if the right coordinate system is chosen. Orthogonal coordinate systems means the coordinates are mutually perpendicular.

- **Cartesian coordinates**

We've already worked with this one in the previous section. Its vector properties are summarized in Table 3.1 in Ulaby.

Let's look at some differential quantities, illustrated in Fig. 7:

Length:

$$d\mathbf{l} = \hat{\mathbf{x}}\, dl_x + \hat{\mathbf{y}}\, dl_y + \hat{\mathbf{z}}\, dl_z = \hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz \qquad (38)$$

Surface: This vector has magnitude equal to the product of two differential lengths and the direction is along the third axis.

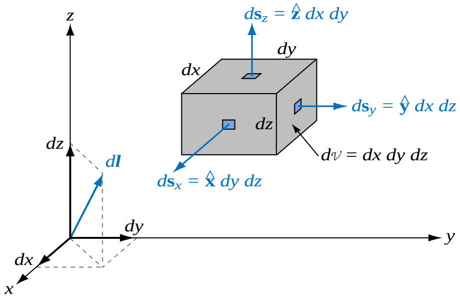


Figure 7: Differential length, area, and volume in Cartesian coordinates.

$$\begin{aligned} d\mathbf{s}_x &= \hat{\mathbf{x}}\, dl_y\, dl_z = \hat{\mathbf{x}}\, dy\, dz \quad (y\text{-}z \text{ plane}) && (39) \\ d\mathbf{s}_y &= \hat{\mathbf{y}}\, dx\, dz \quad (x\text{-}z \text{ plane}) && (40) \\ d\mathbf{s}_z &= \hat{\mathbf{z}}\, dx\, dy \quad (x\text{-}y \text{ plane}) && (41) \end{aligned}$$

Volume : Scalar and equal to the product of the three differential lengths,

$$d\nu = dx\, dy\, dz \tag{42}$$

- **Cylindrical coordinates**

What are Cylindrical coordinates used for? Think about coaxial lines. Cylindrical coordinates use three variables: r, ϕ, z, shown in Fig. 8.

- $r =$ radial distance in the $x - y$ plane. Range of values: $0 \leq r < \infty$.
- $\phi =$ azimuth angle measured from the positive $x-$axis. Range of values: $0 \leq \phi < 2\pi$
- z — same as Cartesian system. Range of values: $-\infty < z < \infty$

Point P is located at the intersection of:

1. Cylindrical surface defined by $r = r_1$,
2. Vertical half-plane defined by $\phi = \phi_1$
3. Horizontal plane defined by $z = z_1$

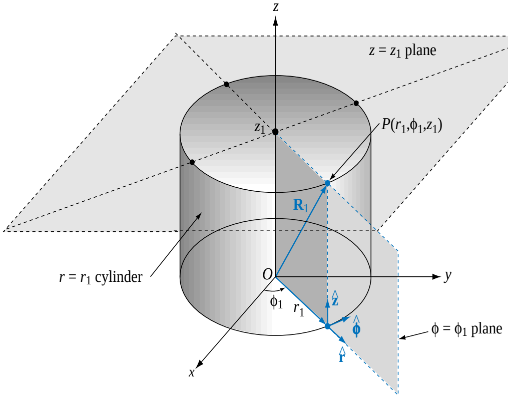


Figure 8: Point $P(r_1, \phi_1, z_1)$ in cylindrical coordinates; r_1 is the radial distance from the origin in the x-y plane, ϕ_1 is the angle in the x-y plane measured from the x-axis toward the y-axis, and z_1 is the vertical distance from the x-y plane.

Mutually perpendicular base vectors:

- $\hat{\mathbf{r}}$ — points away from the origin along r
- $\hat{\phi}$ — pointing tangentially to the cylindrical surface
- $\hat{\mathbf{z}}$ — points along vertical axis.

Some properties:

- $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\phi} \cdot \hat{\phi} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ and cross product with itself $= 0$.
- Base unit vectors obey right-hand cyclic relations

$$\hat{\mathbf{r}} \times \hat{\phi} = \hat{\mathbf{z}}, \quad \hat{\phi} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\phi} \tag{43}$$

- Components of a vector are expressed as

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{r}}A_r + \hat{\phi}A_\phi + \hat{\mathbf{z}}A_z \tag{44}$$

  where components are along their respective axis directions.

- Magnitude of a vector is obtained from

$$|\mathbf{A}| = \sqrt[+]{A \cdot A} = \sqrt[+]{A_r^2 + A_\phi^2 + A_z^2} \tag{45}$$

Look at Fig. 8: that position vector has components only along r and z.

$$\mathbf{R}_1 = \overrightarrow{O\,P} = \hat{\mathbf{r}} r_1 + \hat{\mathbf{z}} z_1 \tag{46}$$

and its dependence on ϕ_1 is only implicit through $\hat{\mathbf{r}}$.

How about differential elements (shown in Fig. 9):

Length: along axis we have:

$$dl_r = dr, \quad dl_\phi = r\, d\phi, \quad dl_z = dz \tag{47}$$

and in general:

$$d\mathbf{l} = \hat{\mathbf{r}}\, dl_r + \hat{\phi}\, dl_\phi + \hat{\mathbf{z}}\, dl_z = \hat{\mathbf{r}}\, dr + \hat{\phi} r\, d\phi + \hat{\mathbf{z}}\, dz \tag{48}$$

Surface: different surfaces:

$$d\mathbf{s}_r = \hat{\mathbf{r}}\, dl_\phi\, dl_z = \hat{\mathbf{r}} r\, d\phi\, dz \quad (\phi\text{-}z \text{ cylindrical surface}) \tag{49}$$

$$d\mathbf{s}_\phi = \hat{\phi}\, dl_r\, dl_z = \hat{\phi}\, dr\, dz \quad (r\text{-}z \text{ plane}) \tag{50}$$

$$d\mathbf{s}_z = \hat{\mathbf{z}}\, dl_r\, dl_\phi = \hat{\mathbf{z}} r\, dr\, d\phi \quad (r\text{-}\phi \text{ plane}) \tag{51}$$

Volume:

$$d\nu = dl_r\, dl_\phi\, dl_z = r\, dr\, d\phi\, dz \tag{52}$$

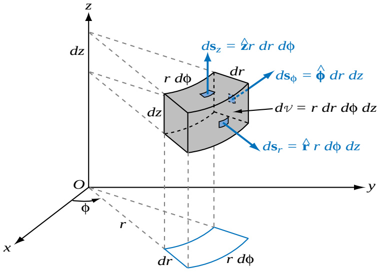


Figure 9: Differential areas and volume in cylindrical coordinates.

• Spherical Coordinates

Position specified by variables R, θ, ϕ, shown in Fig. 10.

- Range coordinate R. Range of values: $0 \leq R < \infty$.
- Zenith angle θ, measured from the positive $z-$ axis; it describes a conical surface with apex at the origin. Range of values: $0 \leq \theta \leq \pi$.
- Azimuth angle — same as in cylindrical system. Range of values: $0 \leq \phi < 2\pi$

Some properties:

- Right-hand cyclic relations are:

$$\hat{\mathbf{R}} \times \hat{\theta} = \hat{\phi}, \quad \hat{\theta} \times \hat{\phi} = \hat{\mathbf{R}}, \quad \hat{\phi} \times \hat{\mathbf{R}} = \hat{\theta} \tag{53}$$

- Vector components are written:

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{R}} A_R + \hat{\theta} A_\theta + \hat{\phi} A_\phi \tag{54}$$

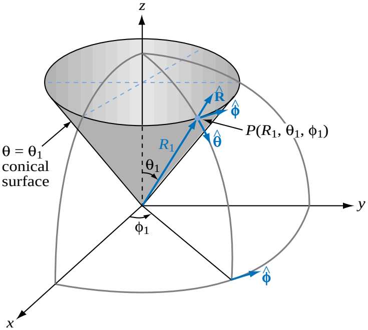


Figure 10: Point $P(R_1, \theta_1, \phi_1)$ in spherical coordinates.

- The vector magnitude:

$$|\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}} = \sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2} \tag{55}$$

- The position vector of the point $P(R_1, \theta_1, \phi_1)$,

$$\mathbf{R}_1 = \overrightarrow{O\,P} = \hat{\mathbf{R}} R_1 \tag{56}$$

  but remember that $\hat{\mathbf{R}}$ is implicitly dependent on θ_1, ϕ_1.

The differential lengths (shown in fig. 11):

Length:

$$dl_R = dR, \quad dl_\theta = R\,d\theta, \quad dl_\phi = R\sin\theta\,d\phi \tag{57}$$

$$\begin{aligned} d\mathbf{l} &= \hat{\mathbf{R}}\,dl_R + \hat{\theta}\,dl_\theta + \hat{\phi}\,dl_\phi \\ &= \hat{\mathbf{R}}\,dR + \hat{\theta}R\,d\theta + \hat{\phi}R\sin\theta\,d\phi \end{aligned} \tag{58}$$

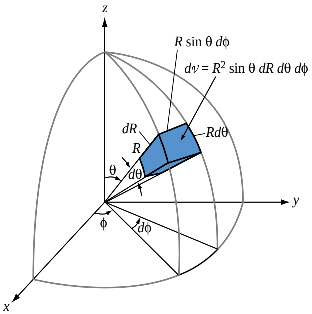


Figure 11: Differential volume in spherical coordinates.

Surface:

$$d\mathbf{s}_R = \hat{\mathbf{R}}\,dl_\theta\,dl_\phi = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi \quad (\theta\text{-}\phi \text{ spherical surface}) \tag{59}$$

$$d\mathbf{s}_\theta = \hat{\theta}\,dl_R\,dl_\phi = \hat{\theta}R\sin\theta\,dR\,d\phi \quad (R\text{-}\phi \text{ conical surface}) \tag{60}$$

$$d\mathbf{s}_\phi = \hat{\phi}\,dl_R\,dl_\theta = \hat{\phi}R\,dR\,d\theta \quad (R\text{-}\theta \text{ plane}) \tag{61}$$

Volume:

$$d\nu = dl_R\,dl_\theta\,dl_\phi = R^2\sin\theta\,dR\,d\theta\,d\phi \tag{62}$$

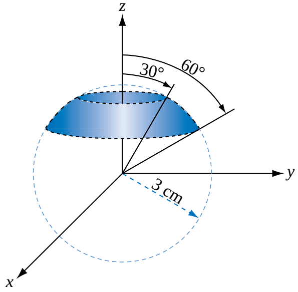


Figure 12: Spherical strip of Example 3-5.

3.3. Coordinate transformations

Positions and vectors are the same no matter what coordinate system we use $\Rightarrow$ we can transform one set of coordinates to another.

- **Cartesian to cylindrical**

Take point P in Fig. 13. Its Cartesian coordinates are (x, y, z), and cylindrical are (r, ϕ, z). Note that z coordinate is shared; the other two can be determined from the geometry, so that cartesian to cylindrical coordinate transformation is:

$$r = \sqrt[+]{x^2 + y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{63}$$

and cylindrical to cartesian,

$$x = r\cos\phi, \quad y = r\sin\phi \tag{64}$$

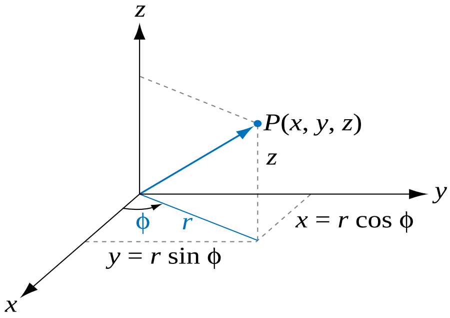


Figure 13: Interrelationships between Cartesian coordinates (x, y, z) and cylindrical coordinates (r, ϕ, z).

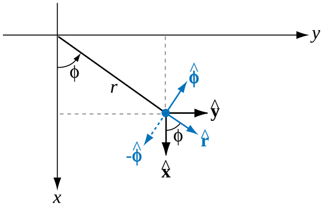


Figure 14: Interrelationships between base vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ and $(\hat{\mathbf{r}}, \hat{\phi})$.

How about the relationship between base vectors? Refer to Fig. 14. The procedure involves realizing, from the geometry that,

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \cos\phi, \qquad \hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin\phi \tag{65}$$

$$\hat{\phi} \cdot \hat{\mathbf{x}} = -\sin\phi, \qquad \hat{\phi} \cdot \hat{\mathbf{y}} = \cos\phi \tag{66}$$

(Maybe easier to see if we let $\alpha = \pi/2 - \phi$ so $\phi = \pi/2 - \alpha$ and $\cos(\alpha) = \sin(\pi/2 - \alpha) = \sin(\phi)$)

We can write $\hat{\mathbf{r}}$ in terms of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}a + \hat{\mathbf{y}}b \tag{67}$$

where we don't yet know the values for a and b. Use the dot product to solve for a,

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}a + \hat{\mathbf{y}} \cdot \hat{\mathbf{x}}b = a \tag{68}$$

and recall, $\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \cos\phi$ so, $a = \cos\phi$. Similarly, $b = \hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin\phi$. The same can be done for $\hat{\phi}$ leading to,

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \tag{69}$$

$$\hat{\phi} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \tag{70}$$

remember that $\hat{\mathbf{z}}$ is the same in cylindrical and cartesian. For the inverse relations we can solve the above simultaneously to get,

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\phi}\sin\phi \tag{71}$$

$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\phi}\cos\phi \tag{72}$$

These are useful for converting vectors from one coordinate system to another. Remember that in Cartesian coordinates is $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$and in cylindrical $\mathbf{A} = \hat{\mathbf{r}}A_r + \hat{\phi}A_\phi + \hat{\mathbf{z}}A_z$. Using,

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\phi}\sin\phi \quad (73)$$

$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\phi}\cos\phi \quad (74)$$

we can write $\mathbf{A}$ as,

$$\mathbf{A} = A_x\left[\hat{\mathbf{r}}\cos\phi - \hat{\phi}\sin\phi\right] + A_y\left[\hat{\mathbf{r}}\sin\phi + \hat{\phi}\cos\phi\right] + \hat{\mathbf{z}}A_z \quad (75)$$

and collect the terms for A_r and A_ϕ,

$$A_r = A_x\cos\phi + A_y\sin\phi \quad (76)$$

$$A_\phi = -A_x\sin\phi + A_y\cos\phi \quad (77)$$

and conversely, using

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \quad (78)$$

$$\hat{\phi} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \quad (79)$$

$$A_x = A_r \cos\phi - A_\phi \sin\phi \quad (80)$$

$$A_y = A_r \sin\phi + A_\phi \cos\phi \quad (81)$$

- **Cartesian to spherical**

Use Fig. 15 as a starting point. From it we obtain:

$$R = \sqrt[+]{x^2 + y^2 + z^2} \tag{82}$$

$$\theta = \tan^{-1}\left[\frac{\sqrt[+]{x^2 + y^2}}{z}\right] \tag{83}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{84}$$

and inversely (recognizing that $r = R\sin\theta$),

$$x = R\sin\theta\cos\phi \tag{85}$$

$$y = R\sin\theta\sin\phi \tag{86}$$

$$z = R\cos\theta \tag{87}$$

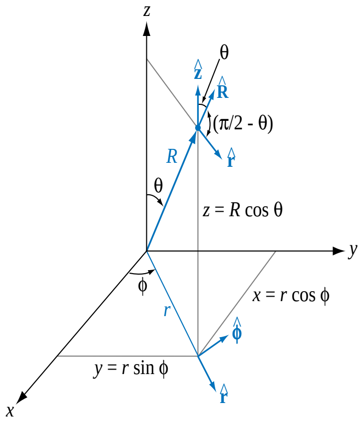


Figure 15: Interrelationships between (x, y, z) and (R, θ, ϕ).

$\hat{\mathbf{R}}$ is always some combination of $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ so,

$$\hat{\mathbf{R}} = \hat{\mathbf{r}}a + \hat{\mathbf{z}}b \tag{88}$$

$$\hat{\mathbf{R}} \cdot \hat{\mathbf{r}} = a \tag{89}$$

$$\hat{\mathbf{R}} \cdot \hat{\mathbf{z}} = b \tag{90}$$

Also, note that $\hat{\mathbf{R}} \cdot \hat{\mathbf{z}} = \cos\theta = b$ and $\hat{\mathbf{R}} \cdot \hat{\mathbf{r}} = \cos(\pi/2 - \theta) = \sin\theta = a$. Also, recall,

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \tag{91}$$

So we obtain the following base vector transformation:

$$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta \tag{92}$$

The other base vectors transform similarly to give:

$$\hat{\mathbf{R}} = \hat{\mathbf{x}} \sin\theta \cos\phi + \hat{\mathbf{y}} \sin\theta \sin\phi + \hat{\mathbf{z}} \cos\theta \quad (93)$$

$$\hat{\theta} = \hat{\mathbf{x}} \cos\theta \cos\phi + \hat{\mathbf{y}} \cos\theta \sin\phi - \hat{\mathbf{z}} \sin\theta \quad (94)$$

$$\hat{\phi} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi \quad (95)$$

The inverse operations are obtained from:

$$\begin{aligned} \hat{\mathbf{x}} &= \hat{\mathbf{R}} \sin\theta \cos\phi + \hat{\theta} \cos\theta \cos\phi - \hat{\phi} \sin\phi && (96) \\ \hat{\mathbf{y}} &= \hat{\mathbf{R}} \sin\theta \sin\phi + \hat{\theta} \cos\theta \sin\phi + \hat{\phi} \cos\phi && (97) \\ \hat{\mathbf{z}} &= \hat{\mathbf{R}} \cos\theta - \hat{\theta} \sin\theta && (98) \end{aligned}$$

To transform components, just replace unit vectors with their respective component values, i.e. $\hat{\mathbf{x}} \to A_x$, $\hat{\mathbf{R}} \to A_R$.

• Distance between two points

We know how to find distance between two points in Cartesian system:

$$\begin{aligned} d &= |\mathbf{R}_{12}| \\ &= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} \end{aligned} \tag{99}$$

Utilizing the transformations in eq. 64, for cylindrical coordinates this becomes

$$\begin{aligned} d &= [(r_2 \cos\phi_2 - r_1 \cos\phi_1)^2 \\ &\quad +(r_2 \sin\phi_2 - r_1 \sin\phi_1)^2 + (z_2 - z_1)^2]^{1/2} \\ &= [r_2^2 + r_1^2 - 2 r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2]^{1/2} \\ &\quad \text{(cylindrical)} \end{aligned} \tag{100}$$

Similarly, for spherical coordinates use eq. 85 -87

$$\begin{aligned} d &= \{R_2^2 + R_1^2 - 2R_1 R_2 [\cos\theta_2 \cos\theta_1 \\ &\quad + \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1)]\}^{1/2} \\ &\quad \text{(spherical)} \end{aligned} \tag{101}$$

3.4. Gradient of a scalar field

Things are simple if we have a scalar that depends on only one quantity ⇒ finding the rate of change is simply $df(z)/dz$. What do we do about 3-D (scalars and vectors)? For 3-D we can do partial derivatives, but how do we combine them? We use gradient, divergence and curl operators. Gradient operates on scalars, the others operates on vectors.

Take temperature $T_1(x, y, z)$ as an example, shown in Fig. 16.

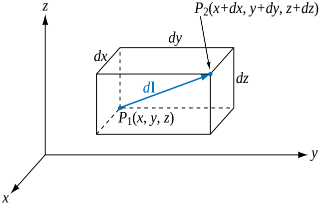


Figure 16: Differential distance vector d**l** between points P_1 and P_2.

- The differential distance $d\mathbf{l}$ has components

$$d\mathbf{l} = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz \tag{102}$$

- The differential temperature $dT = T_2 - T_1$ is

$$dT = \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz \tag{103}$$

- We have by definition, $dx = \hat{\mathbf{x}} \cdot d\mathbf{l}$, $dy = \hat{\mathbf{y}} \cdot d\mathbf{l}$ and $dz = \hat{\mathbf{z}} \cdot d\mathbf{l}$ to get,

$$\begin{aligned} dT &= \hat{\mathbf{x}}\frac{\partial T}{\partial x} \cdot d\mathbf{l} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} \cdot d\mathbf{l} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \cdot d\mathbf{l} \\ &= \left[\hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\right] \cdot d\mathbf{l} \end{aligned} \tag{104}$$

which is the change in temperature corresponding to a vector change in position $d\mathbf{l}$.

- This is called the gradient of T or grad T or ∇T.

$$\nabla T = grad\ T \triangleq \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \tag{105}$$

- We can now plug this into eq. 104

$$dT = \nabla T \cdot d\mathbf{l} \tag{106}$$

where ∇ is called del or gradient operator, defined as

$$\nabla \triangleq \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \quad \text{(Cartesian)} \tag{107}$$

- Quote from Ulaby (in blue print in the text): "whereas the gradient operator has no physical meaning by itself, it attains a physical meaning once it operates on a scalar physical quantity, and the result of the operation is a vector whose magnitude is equal to the maximum rate of change of the physical quantity per unit distance and whose direction is along the direction of maximum increase."

- Define a unit vector in the direction of $d\mathbf{l}$ as $d\mathbf{l} = \hat{\mathbf{a}}_l dl$ so that the **directional derivative** of T along direction of $\hat{\mathbf{a}}_l$ is

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l \tag{108}$$

- Finally, if ∇T is a known 3-D function, the difference $T_2 - T_1$ (see Fig. 16) is calculated from,

$$T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot d\mathbf{l} \tag{109}$$

- **Gradient operator in cylindrical and spherical coordinates**

We derived the previous using Cartesian coordinates but we should have an equivalent operation in any orthogonal coordinate system, namely, cylindrical and spherical. So what do we do in cylindrical and spherical coordinate systems? We have to express ∇ in these coordinate systems. Recall that in the cylindrical system we get,

$$r = \sqrt[+]{x^2 + y^2}, \quad \tan\phi = \left(\frac{y}{x}\right) \tag{110}$$

and we can use the chain rule,

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial x} \tag{111}$$

and the derivatives,

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos\phi \tag{112}$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r} \tag{113}$$

Note, this one is a bit trickier— you have to remember some calculus. $\phi = \tan^{-1}(y/x)$ and $\frac{d\tan^{-1}(y/x)}{dx} = \frac{d(y/x)/dx}{1+(y/x)^2}$, so you get $\frac{-y/x^2}{1+(y/x)^2} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2} = -\sin\phi/r$. So we get,

$$\frac{\partial T}{\partial x} = \cos\phi\frac{\partial T}{\partial r} - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi} \tag{114}$$

which can be used in,

$$\nabla T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \tag{115}$$

We get a simlar expression for for $\partial T/\partial y$. (what about $\partial T/\partial z$?). We also need to express unit vectors; for that use,

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\phi}\sin\phi \tag{116}$$

$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\phi}\cos\phi \tag{117}$$

So that we get,

$$\nabla T = \hat{\mathbf{r}}\frac{\partial T}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial T}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \tag{118}$$

or

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \quad \text{(cylindrical)} \tag{119}$$

In spherical coordinates we get

$$\nabla = \hat{\mathbf{R}}\frac{\partial}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi} \quad \text{(spherical)} \tag{120}$$

- **Properties of the gradient operator**

$$(1) \quad \nabla(U+V) = \nabla U + \nabla V \tag{121}$$

$$(2) \quad \nabla(U\,V) = U\,\nabla V + V\,\nabla U \tag{122}$$

$$(3) \quad \nabla V^n = nV^{n-1}\,\nabla V \quad \text{for any } n \tag{123}$$

Note that gradient of a vector is meaningless.

3.5. Divergence of a vector field

First a little background. We've already seen how electric charge introduces an electric field around it, as illustrated in Fig. 17. This **vector** field is represented by field lines (little arrows). The field itself does not move but it can move charge introduced into that field, so we think of field lines as flux lines and define their flux density as amount of outward flux crossing a unit surface ds, i.e.

$$\text{Flux density of } \mathbf{E} = \frac{\mathbf{E} \cdot d\mathbf{s}}{|d\mathbf{s}|} = \frac{\mathbf{E} \cdot \hat{\mathbf{n}}\, ds}{ds} \tag{124}$$

where $\hat{\mathbf{n}}$ is the outward surface normal of $d\mathbf{s}$.

The total flux crossing a closed surface is simply a surface integral

$$\text{Total flux } = \oint_S \mathbf{E} \cdot d\mathbf{s} \tag{125}$$

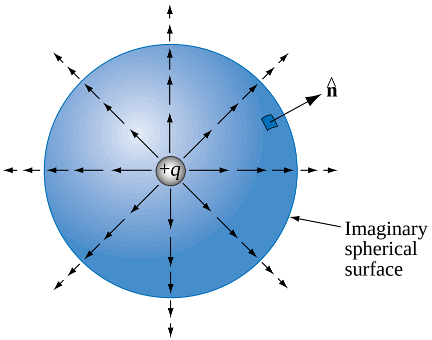


Figure 17: Flux lines of the electric field **E** due to a positive charge q.

Let's now look at Fig. 18 and try to calculate the total flux.

- We start with a parallelepiped such as the cube shown
- There are six faces — we sum up fluxes over all of them. Start by defining **E**.

$$\mathbf{E} = \hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y + \hat{\mathbf{z}}E_z \tag{126}$$

- Note that the outward normal vector on surface 1 is in the negative x direction, i.e. $\hat{\mathbf{n}}_\mathbf{1} = -\hat{\mathbf{x}}$, so that

$$\begin{aligned} F_1 &= \int_{\text{Face 1}} \mathbf{E} \cdot \hat{\mathbf{n}}_1 \, ds \\ &= \int_{\text{Face 1}} (\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y + \hat{\mathbf{z}}E_z) \cdot (-\hat{\mathbf{x}}) \, dy \, dz \\ &= -E_x(1) \; \Delta y \, \Delta z \end{aligned} \tag{127}$$

where we've assumed $E(x)$ to be constant over the face and equal to the value at the center of the face.

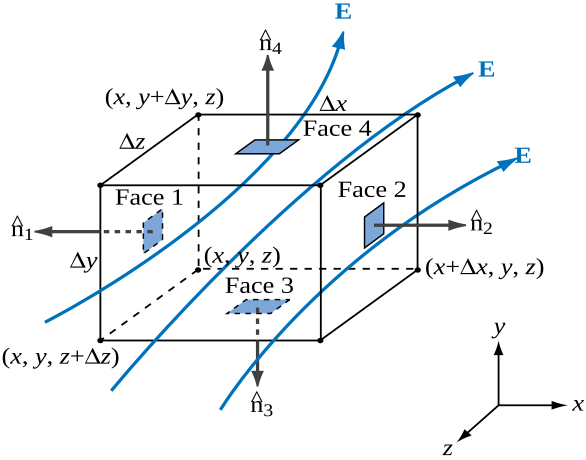


Figure 18: Flux lines of a vector field $\mathbf{E}$ passing through a differential rectangular parallelepiped of volume $\Delta v = \Delta x\,\Delta y\,\Delta z$.

- On face 2 we get

$$F_2 = E_x(2)\ \Delta y\, \Delta z \tag{128}$$

- By using Taylor's expansion, we can express (approximately) the value on one face in terms of the other face value, i.e.

$$E_x(2) = E_x(1) + \frac{\partial E_x}{\partial x}\,\Delta x \tag{129}$$

so that

$$F_2 = \left[E_x(1) + \frac{\partial E_x}{\partial x}\,\Delta x\right]\,\Delta y\, \Delta z \tag{130}$$

and

$$F_1 + F_2 = \frac{\partial E_x}{\partial x}\,\Delta x\,\Delta y\,\Delta z \tag{131}$$

$$F_3 + F_4 = \frac{\partial E_y}{\partial y}\,\Delta x\,\Delta y\,\Delta z \tag{132}$$

$$F_5 + F_6 = \frac{\partial E_z}{\partial z}\,\Delta x\,\Delta y\,\Delta z \tag{133}$$

- The grand total is then:

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$
$$= (\text{div } \mathbf{E}) \Delta \nu \quad (134)$$

where $\Delta\nu$ is the volume and div $\mathbf{E}$ is a differential function called **divergence** of $\mathbf{E}$ and is defined as

$$\text{div } \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \quad (135)$$

- By using the usual trick of reducing the dimensions to zero, we get divergence of $\mathbf{E}$ at a point

$$\text{div } \mathbf{E} \triangleq \lim_{\Delta\nu \to 0} \frac{\oint_S \mathbf{E} \cdot d\mathbf{s}}{\Delta\nu} \quad (136)$$

- In alternative notation,

$$\nabla \cdot \mathbf{E} \triangleq \text{div } \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \quad (137)$$

Interpretation: from,

$$\text{div } \mathbf{E} \triangleq \lim_{\Delta\nu\to 0} \frac{\oint_S \mathbf{E}\cdot d\mathbf{s}}{\Delta\nu} \tag{138}$$

the field **E** has positive divergence if the net flux out of surface S is positive ⇒ some source of flux is present within the volume. If it is negative ⇒ there is a sink present.

- If **E** is uniform ⇒ the same amount of flux enters and leaves ⇒ divergence is zero (divergenceless field).
- Divergence operates only on vectors and the result is scalar. It can also be applied in cylindrical and spherical systems.
- Divergence is distributive

$$\nabla\cdot(\mathbf{E}_1+\mathbf{E}_2) = \nabla\cdot\mathbf{E}_1 + \nabla\cdot\mathbf{E}_2 \tag{139}$$

- If $\nabla\cdot\mathbf{E} = 0$ ⇒ solenoidal field.

• Divergence theorem

Extending

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$
$$= (\text{div } \mathbf{E}) \Delta \nu \quad (140)$$

from differential volume $\Delta \nu$ to a volume integral,

$$\int_\nu \nabla \cdot \mathbf{E} d\nu = \oint_S \mathbf{E} \cdot d\mathbf{s} \quad \text{(divergence theorem)} \quad (141)$$

which is known as the divergence theorem. The closed surface in the integral on the right is the surface that bounds the volume that is integrated over on the left.

3.6. Curl of a vector field

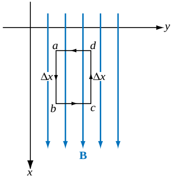
So, why do we need yet another operator? There is an additional property of fields called circulation, which is defined as a line integral of the field around a closed contour.

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l} \tag{142}$$

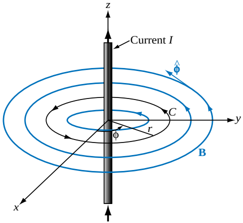
To illustrate, have a look at Fig. 19. For case a) we observe that the circulation$= 0$, or mathematically,

$$\begin{aligned} \text{Circulation} &= \int_a^b \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{x}}\, dx + \int_b^c \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{y}}\, dy \\ &+ \int_c^d \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{x}}\, dx + \int_d^a \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{y}}\, dy \\ &= B_0\, \Delta x - B_0\, \Delta x = 0 \end{aligned} \tag{143}$$

where $\Delta x = b - a = c - d$ and recall, $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$.



(a) Uniform field



(b) Azimuthal field

Figure 19: Circulation is zero for the uniform field in (a), but it is not zero for the azimuthal field in (b).

$\Rightarrow$ **circulation of a uniform field is zero**.

Case b) shows magnetic filed induced by current I (what is the best coordinate system to use?). Field lines are concentric circles around the current source.

$$\mathbf{B} = \hat{\phi}\,\frac{\mu_0 I}{2\pi r} \tag{144}$$

Suppose we have a circular contour of radius r then the differential length vector is $d\mathbf{l} = \hat{\phi} r d\phi$ so that circulation of $\mathbf{B}$ around the contour is

$$\begin{aligned} \text{Circulation} &= \oint_C \mathbf{B} \cdot d\mathbf{l} \\ &= \int_0^{2\pi} \hat{\phi}\,\frac{\mu_0 I}{2\pi r} \cdot \hat{\phi} r\, d\phi = \mu_0 I \end{aligned} \tag{145}$$

- This circulation is not zero, but what about other contours? Any contour in planes that are perpendicular to the $x-y$ plane would have $= 0$ (because the differential length would not have a ϕ component).

- Also, the direction of contour determines the sign of circulation
- The curl operator is used to describe the circulation of a vector field. It is denoted as curl $\mathbf{B}$ or $\nabla \times \mathbf{B}$.

$$\nabla \times \mathbf{B} = curl\ \mathbf{B} \triangleq \lim_{\triangle s \to 0} \frac{1}{\triangle s} \left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{max} \tag{146}$$

- curl $\mathbf{B}$ is the circulation of $\mathbf{B}$ per unit area, with the area Δs of the contour C being oriented such that the circulation is maximum.
- The direction of curl $\mathbf{B}$ is $\hat{\mathbf{n}}$, which is normal to Δs using the right hand rule.

- So, how do we get something out of eq. 146? In Cartesian coordinates,

$$\mathbf{B} = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z \tag{147}$$

we get (omitting the long derivation),

$$\begin{aligned} \nabla \times \mathbf{B} &= \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) \\ &= +\,\hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} \end{aligned} \tag{148}$$

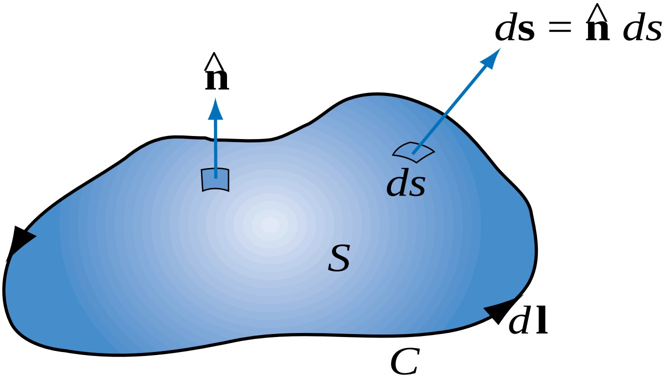


Figure 20: The direction of the unit vector $\hat{x}$ is along the thumb when the other four fingers of the right hand follow d **l**.

- **Vector identities involving curl**

$$(1) \quad \nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \tag{149}$$

$$(2) \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad \text{for any vector } \mathbf{A} \tag{150}$$

$$(3) \quad \nabla \times (\nabla V) = 0 \quad \text{for any scalar function V} \tag{151}$$

- **Stoke's theorem**

Using this theorem we can convert the surface integral of the curl of a vector over an open surface S into a line integral of the vector along the contour C bounding the surface S.

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l} \quad \text{(Stokes's theorem)} \tag{152}$$

If $\nabla \times \mathbf{B} = 0$ the field is said to be conservative or irrotational

3.7. Laplacian operator

Another combination of operators: divergence of a gradient of a scalar (or vector). In Cartesian coordinates

$$\begin{aligned} \nabla V &= \hat{\mathbf{x}} \frac{\partial V}{\partial x} + \hat{\mathbf{y}} \frac{\partial V}{\partial y} + \hat{\mathbf{z}} \frac{\partial V}{\partial z} \\ &= \hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z = \mathbf{A} \end{aligned} \tag{153}$$

and divergence of it is

$$\begin{aligned} \nabla \cdot (\nabla V) = \nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \end{aligned} \tag{154}$$

and we call it Laplacian of V and is denoted by $\nabla^2 V$

$$\nabla^2 V \triangleq \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{155}$$

which is a scalar

We can also define a Laplacian of a vector

$$\mathbf{E} = \hat{\mathbf{x}} E_x + \hat{\mathbf{y}} E_y + \hat{\mathbf{z}} E_z \tag{156}$$

such that

$$\begin{aligned} \nabla^2 \mathbf{E} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E} \\ &= \hat{\mathbf{x}}\, \nabla^2 E_x + \hat{\mathbf{y}}\, \nabla^2 E_y + \hat{\mathbf{z}}\, \nabla^2 E_z \end{aligned} \tag{157}$$

or, in Cartesian coordinates the Laplacian of a vector is a vector whose components are equal to the Laplacians of the vector components. The following also holds:

$$\nabla^2 \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}) \tag{158}$$