Contents

2. Transmission lines

2.1. Transmission Lines: General Considerations

What is a transmission line? For us it will be a pair of wires (or a waveguide) used to **guide** electromagnetic signals, e.g. telephone wires, coaxial cables, optical fibers etc. Schematically, Fig. 1 presents a transmission line as a two port circuit with input source and output load connected. Various loads are possible.

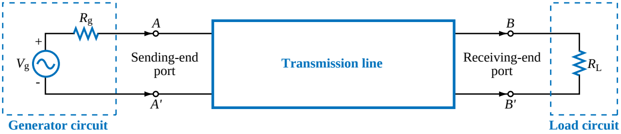


Figure 1: Transmission line as a black box.

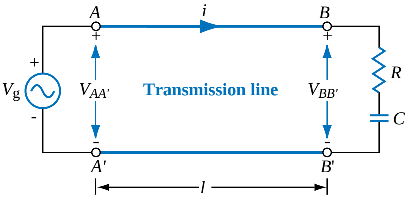


Figure 2: Transmission line in a circuit.

- **Wavelength and transmission lines**

Why didn't we worry about transmission lines when studying basic circuits? Check out Fig. 2: simple AC generator connected to R-C load via a transmission line. Can we just take out the transmission line? It depends...

- The generator sends a cosinusoidal signal $V_{AA'} = V_g(t) = V_0 \cos \omega t$.

- Assume that the signal (current or voltage) travels with speed of light in vacuum $c = 3 \times 10^8$ m/s
- How much is the signal delayed going from AA' to BB'?
- If there are no ohmic losses

$$V_{BB'}(t) = V_{AA'}(t - l/c) = V_0 \cos[\omega(t - l/c)](\text{V}) \tag{1}$$

- Take a wire length of $l = 5$ cm . Set the time to $t = 0$ s. For $f = 1$ kHz, this case gives $V_{BB'} = 0.999 \ldots V_0$, i.e. $V_{BB'}$ and $V_{AA'}$ are indistinguishable.
- In the second case take $l = 20$km. This gives $V_{BB'} = 0.91 V_0$, i.e. quite a difference between $V_{BB'}$ and $V_{AA'}$.
- Where is this coming from? The term $\omega l/c$. That can be re-expressed using $u_p = c$ (here) so $c = f\lambda$ (m/s), so that

$$\frac{\omega l}{c} = \frac{2\pi f l}{c} = 2\pi \frac{l}{\lambda} \text{ radians} \tag{2}$$

- If l/λ is small $\Rightarrow$ transmission line effects negligible
- If $l/\lambda \gtrsim 0.01 \Rightarrow$ transmission line effects must be considered
- In addition to time delay (shift), we also need to take into account **reflections**, power loss and dispersion.
- What's dispersion? Wave velocity is not constant but is a function of frequency. Figure 3 shows dispersive effects. Could affect operation of digital circuits ("signal integrity").

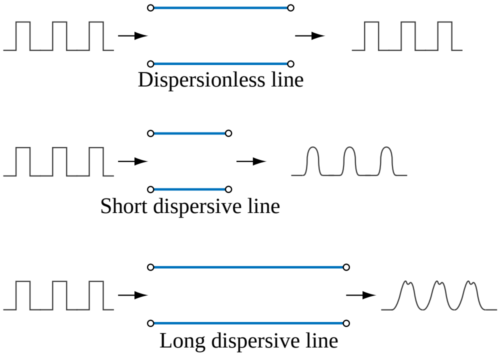


Figure 3: Transmission line as a source of distortion.

• Propagation modes

Have a look at a few transmission lines in Fig. 4

- There are two basic types: TEM and higher order
- TEM = **T**ransverse **E**lectro**M**agnetic lines. Electric and magnetic fields are transverse to the direction of propagation. Coaxial lines are one type. Microstrip not exactly TEM but can be a close approximation to a TEM waveguide.
- Metalic waveguides (e.g. rectangular) and fiber optic lines are typical for higher order transmission lines. These have at least one component of the E or H field that points in the direction of waveguide propagation.

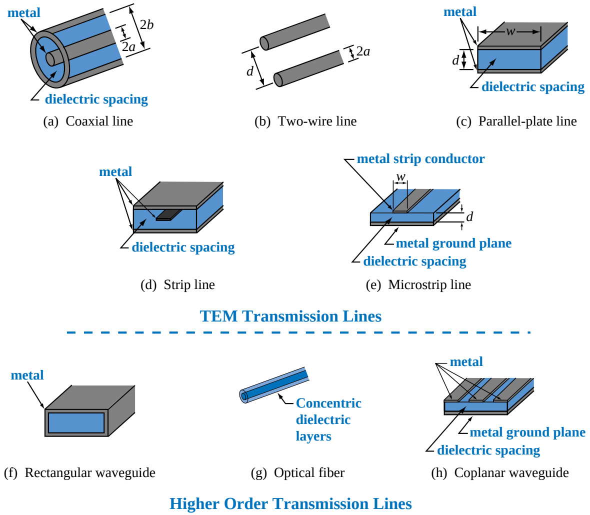


Figure 4: Various transmission lines.

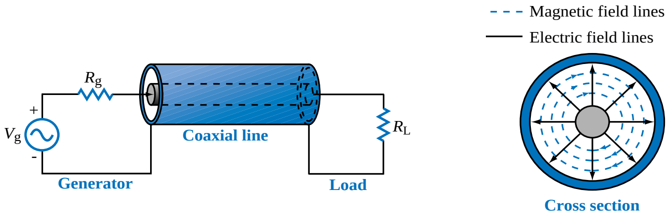


Figure 5: Details of coaxial transmission line.

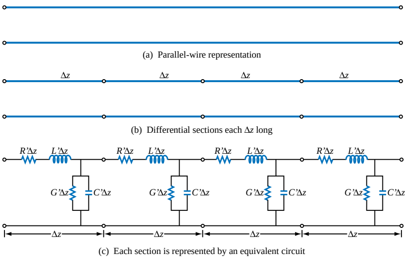


Figure 6: Equivalent circuit representation of a TEM transmission line.

2.2. Lumped element model

We can represent TEM transmission lines of all kinds by a *parallel wire* configuration as shown in Fig. 6. But how do we describe this transmission lines model? The same figure is a starting point.

Lumped element model:

- Represent the transmission line with equivalent *lumped circuit*, i.e. one consisting of resisitive, inductive and capacitive (R, L and C) components. Start by breaking into sections of length Δz.
- The transmission line parameters in the sections are:
  - R' = resistance of both conductors per unit length Ω/m
  - L' = inductance of both conductors H/m,
  - G' = conductance of insulation medium S/m
  - C' = capacitance of two conductors F/m
- The same model is applicable to all TEM-mode wave propagation transmission lines.
- Note that all parameters are in units/length (the prime is used as a reminder that these are per unit length).

- The values for these parameters differ, depending on specifics of the transmission line. For now, we have to accept these but we will derive the expressions later on.

- The transmission line parameters for coaxial, two wire and parallel plate waveguides are in Table 2-1 from the book, where μ_c, σ_c are magnetic permeability and conductivity of conductors, ϵ, μ, σ are electrical permittivity, magnetic permeability, and electrical conductivity of insulation material between conductors.

This is illustrated for coaxial line in Fig.7. Let's have a look at each transmission line parameters.

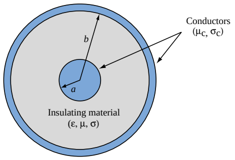


Figure 7: Cross-section of coaxial transmission line.

- R' is resistance of both inner and outer conductors.

$$R' = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) \ (\Omega/\mathrm{m}) \tag{3}$$

- The intrinsic resistance R_s is the surface resistance of conductors. Note the frequency dependence!

$$R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}} \ (\Omega) \tag{4}$$

- What happens when $\sigma_c \to \infty$?
- $R_s \to 0$ and $R' \to 0$.
- For inductance per unit length we have

$$L' = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) \ \text{(H/m)} \tag{5}$$

- G' is shunt conductance, i.e. current flowing between two conductors.

$$G' = \frac{2\pi\sigma}{\ln(b/a)} \ \text{(S/m)} \tag{6}$$

- What happens if we have perfect dielectric material? Then $\sigma = 0$ and therefore $G' = 0$.
- C' is the capacitance between two conductors. In this case

$$C' = \frac{2\pi\epsilon}{\ln(b/a)} \ \text{(F/m)} \tag{7}$$

- Not only coax, but all TEM lines have the following relations:

$$L'C' = \mu\epsilon, \quad \frac{G'}{C'} = \frac{\sigma}{\epsilon} \tag{8}$$

- If the insulating material between the conductors is air the transmission line is called and air line (free space parameters and $\sigma = 0$, $G' = 0$).

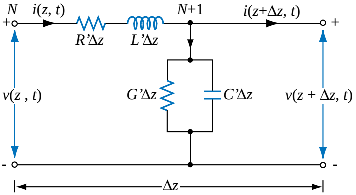


Figure 8: Δz section of a transmission line.

2.3. Transmission line equations

Now that we have an equivalent circuit (model) for transmission lines, what do we do? Start with a segment shown in Fig. 8 and write some Kirchhoff's equations for voltages and currents.

- The voltage drop between nodes N and $N+1$ gives the first equation:

$$v(z,t) - R'\Delta z\ i(z,t) - L'\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0 \quad (9)$$

$$-\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = R'\ i(z,t) + L'\frac{\partial i(z,t)}{\partial t} \quad (10)$$

- If we take the limit as $\Delta z \to 0$ this becomes a differential equation:

$$-\frac{\partial v(z,t)}{\partial z} = R' i(z,t) + L'\frac{\partial i(z,t)}{\partial t} \quad (11)$$

- The sum of the currents at node $N+1$ gives the 2nd equation

$$i(z,t) - i(z+\Delta z,t) - i_G - i_C = 0 \tag{12}$$

Where the current in the resistor is (use Ohm's law $V = IR$ where $R = 1/G$),

$$i_G = \frac{v(z+\Delta z,t)}{1/G'\Delta z} = G'\Delta z \; v(z+\Delta z,t) \tag{13}$$

For the capacitor, start with equation relating voltage, charge and capacitance, $V = Q/C$ or $Q = CV$,

$$i_C = C'\Delta z \frac{dv(z+\Delta z,t)}{dt} \tag{14}$$

Rearrange and divide by Δz,

$$i(z,t) - i(z+\Delta z,t) - G'\Delta z \; v(z+\Delta z,t) - C'\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} = 0 \tag{15}$$

$$-\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z}=G'v(z+\Delta z,t)+C'\frac{\partial v(z+\Delta z,t)}{\partial t} \quad (16)$$

and let Δz approach zero, so we get a partial differential equation,

$$-\frac{\partial i(z,t)}{\partial z}=G'v(z,t)+C'\frac{\partial v(z,t)}{\partial t} \quad (17)$$

(note, typo in book on page 55 should say Eq. 2.15 becomes the second, first-order equation)

- These are transmission line equations a.k.a. *telegrapher's equations* in the time domain:

$$-\frac{\partial v(z,t)}{\partial z}=R'i(z,t)+L'\frac{\partial i(z,t)}{\partial t} \quad (18)$$

$$-\frac{\partial i(z,t)}{\partial z}=G'v(z,t)+C'\frac{\partial v(z,t)}{\partial t} \quad (19)$$

- Further simplification is possible for sinusoidal sources,

$$v(z,t) = \Re\left[\tilde{V}(z)e^{j\omega t}\right] \tag{20}$$

$$i(z,t) = \Re\left[\tilde{I}(z)e^{j\omega t}\right]. \tag{21}$$

- In that case we have a single frequency (time harmonic) and can go to phasor representation:

$$-\frac{\partial v(z,t)}{\partial z} = R' i(z,t) + L' \frac{\partial i(z,t)}{\partial t} \rightarrow -\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\tilde{I}(z) \tag{22}$$

$$-\frac{\partial i(z,t)}{\partial z} = G' v(z,t) + C' \frac{\partial v(z,t)}{\partial t} \rightarrow -\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\tilde{V}(z) \tag{23}$$

These are equations 2.18a and 2.18b in the book.

2.4. Wave propagation on a transmission line

We have the transmission line equations– how do we solve them?

- First combine two first-order coupled equations into two uncoupled second order equations. How? Take d/dz to get:

$$\frac{d}{dz}\frac{d\tilde{V}(z)}{dz} = \frac{d}{dz}\left[(R' + j\omega L')\tilde{I}(z)\right] \tag{24}$$

$$-\frac{d^2\tilde{V}(z)}{dz^2} = (R' + j\omega L')\frac{d\tilde{I}(z)}{dz} \tag{25}$$

On RHS plug in:

$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\tilde{V}(z) \tag{26}$$

$$\frac{d^2\tilde{V}(z)}{dz^2} = (R' + j\omega L')(G' + j\omega C')\tilde{V}(z) \tag{27}$$

$$\frac{d^2\tilde{V}(z)}{dz^2} - (R' + j\omega L')(G' + j\omega C')\tilde{V}(z) = 0 \quad (28)$$

$$\text{use} \quad \gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} \quad (29)$$

$$\Rightarrow \frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0 \quad (30)$$

- Same can be done for the current

$$\frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0 \quad (31)$$

- These equations (30 and 31) are called **wave equations**.
- Parameter γ is called the **complex propagation constant**. It consists of the real part α = **attenuation constant**, and imaginary part β = **phase constant**

$$\alpha = \Re(\gamma) = \Re\left(\sqrt{(R' + j\omega L')(G' + j\omega C')}\right) \text{(Np/m)} \quad (32)$$

$$\beta = \Im(\gamma) = \Im\left(\sqrt{(R' + j\omega L')(G' + j\omega C')}\right) \text{(rad/m)} \quad (33)$$

- Note that we've assumed α and β to be positive. This will turn out to be a necessity for a propagating wave with possibly some decay.

We still don't have a solution, but with equations in the form

$$\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0 \tag{34}$$

$$\frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0 \tag{35}$$

it is easy to see (how?) that the solutions will be exponentials like,

$$\tilde{V}(z) = V_0 e^{\gamma z} \tag{36}$$

But, we can also have a solution with a negative exponent,

$$\tilde{V}(z) = V_0 e^{-\gamma z} \tag{37}$$

So, we can have a total solution that contains both. We use the superscripts $+$ and $-$ on the amplitude to denote waves going in the positive and negative z directions. The solution then looks like:

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \tag{38}$$

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \tag{39}$$

- Each term represents a wave, so that we have two waves, but going in opposite directions!
- Convention has it that $e^{-\gamma z}$ is a wave in $+z$ direction
- Still no solution; how many unknowns?

- We can take the derivative of,

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \tag{40}$$

$$\frac{d\tilde{V}(z)}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} \tag{41}$$

$$-\frac{d\tilde{V}(z)}{dz} = \gamma \left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z} \right] \tag{42}$$

and plug into,

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\tilde{I}(z) \tag{43}$$

$$\gamma \left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z} \right] = (R' + j\omega L')\tilde{I}(z) \tag{44}$$

$$\tilde{I}(z) = \frac{\gamma}{R' + j\omega L'} \left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z} \right] \tag{45}$$

- Does this look familiar? Compare each of the terms of,

$$\tilde{I}(z) = \frac{\gamma}{R' + j\omega L'} \left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z} \right] \tag{46}$$

with,

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \tag{47}$$

And we can see that,

$$I_0^+ = \frac{\gamma V_0^+}{R' + j\omega L'} \tag{48}$$

or,

$$\frac{V_0^+}{I_0^+} = \frac{R' + j\omega L'}{\gamma} = Z_0. \tag{49}$$

Similarly,

$$\frac{-V_0^-}{I_0^-} = \frac{R' + j\omega L'}{\gamma} = Z_0. \tag{50}$$

That is,

$$\frac{V_0^+}{I_0^+} = Z_0 = \frac{-V_0^-}{I_0^-} \tag{51}$$

- The parameter Z_0 is the characteristic impedance of the transmission line, and is given by (recalling the definition of γ),

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} \tag{52}$$

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \ (\Omega) \tag{53}$$

- An important point is that Z_0 is the ratio of the voltage and current amplitudes for each of the propagating waves separately. The **total voltage** $\tilde{V}(z)$ is the sum of the two waves traveling in opposite directions. Therefore, Z_0 is not the ratio of the total voltage and current.

- Note that V_0^+ and V_0^- are *complex* numbers, so each has a magnitude and phase, i.e. $V_0^+ = |V_0^+|e^{j\phi^+}$, $V_0^- = |V_0^-|e^{j\phi^-}$
- With the boundary conditions we will be able to solve for the voltage along the transmission line in the phasor domain. Remember the process for returning back to the time domain.
  - multiply phasor solution by $e^{j\omega t}$
  - take the real part
- The time domain solution will have the form

$$v(z,t) = \Re\left[\left[V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}\right] e^{j\omega t}\right] \tag{54}$$

$$\Re\left[|V_0^+|e^{j\phi^+} e^{-(\alpha+j\beta)z} e^{j\omega t} + |V_0^-|e^{j\phi^-} e^{(\alpha+j\beta)z} e^{j\omega t}\right] \tag{55}$$

Rearrange a little,

$$\Re\left[|V_0^+|e^{-\alpha z} e^{j(\omega t - \beta z + \phi^+)} + |V_0^-|e^{\alpha z} e^{j(\omega t + \beta z + \phi^-)}\right] \tag{56}$$

$$v(z,t) = |V_0^+|e^{-\alpha z}\cos(\omega t - \beta z + \phi^+) + |V_0^-|e^{\alpha z}\cos(\omega t + \beta z + \phi^-) \tag{57}$$

- From before, the first term is $+z$ traveling wave (why?), while the 2nd one is $-z$ traveling.
- What is the phase (propagation) velocity?

$$u_p = f\lambda = \frac{\omega}{\beta} \tag{58}$$

- Important realization: waves traveling in opposite directions on transmission lines form standing waves!

2.5. The lossless microstrip line

- The microstrip line is a type of transmission line for RF and microwave circuits.
- Microwave circuits are found in many applications including cellular communications, wireless networking, satellite communications and radar.
- These transmission lines are easy to fabricate on a circuit board consisting of just a thin copper strip printed on a dielectric substrate that is over a ground plane.
- It is similar to a parallel plate waveguide that supports TEM modes but since it has limited dimensions is only approximately TEM or quasi-TEM.
- There are two geometric parameters- the width of the strip w and the height (thickness) of the dielectric layer h.

- The thickness of the strip is neglected because it is generally much smaller than w.
- We assume the substrate is a perfect dielectric $\sigma = 0$.
- We assume the strip and ground plane are perfect conductors $\sigma \approx \infty$.
- These approximations and assumptions simplify the analysis quite a bit but do not introduce significant error.
- The three parameters that will determine the characteristics of the transmission line are w, h and ϵ.
- With these assumptions the phase speed of the wave is given by,

$$u_p = \frac{c}{\sqrt{\epsilon_r}} \tag{59}$$

with ϵ_r the relative permittivity and c the speed of light in free space.

- Even though the electric field is mostly in the dielectric substrate, some is in the surrounding air. This mixture of where the electric field is can be accounted for by using an effective permittivity ϵ_{eff} which leads to,

$$u_p = \frac{c}{\sqrt{\epsilon_{\text{eff}}}} \tag{60}$$

- Getting the exact effective permittivity gives a complicated expression but we can get a good approximation by curve fitting to this,

$$\epsilon_{\text{eff}} = \frac{\epsilon_r + 1}{2} + \left(\frac{\epsilon_r - 1}{2}\right)\left(1 + \frac{10}{s}\right)^{-xy} \tag{61}$$

where $s = \frac{w}{h}$ and,

$$x = 0.56\left[\frac{\epsilon_r - 0.9}{\epsilon_r + 3}\right]^{0.05} \tag{62}$$

$$y = 1+0.02\ln\left(\frac{s^4 + 3.7\times10^{-4}s^2}{s^4+0.43}\right)+0.05\ln\left(1+1.7\times10^{-4}s^3\right) \tag{63}$$

- The characteristic impedance is given by,

$$Z_0 = \frac{60}{\sqrt{\epsilon_{\text{eff}}}}\ln\left[\frac{6+(2\pi-6)e^{-t}}{s}+\sqrt{1+\frac{4}{s^2}}\right] \tag{64}$$

with

$$t = \left(\frac{30.67}{s}\right)^{0.75} \tag{65}$$

- The figure shows the relationship between Z_0 and s for various dielectric materials.

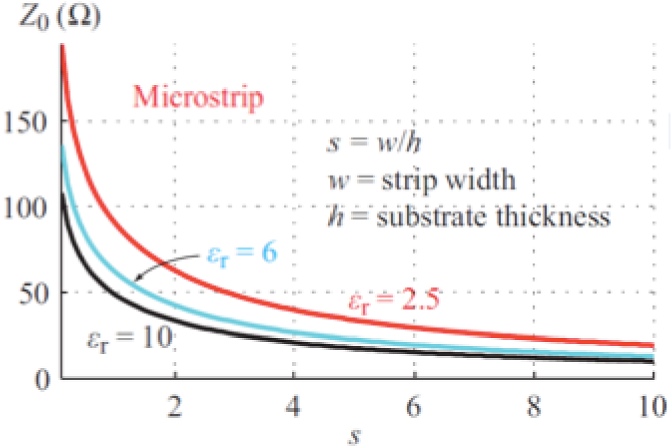


Figure 9: Plots of Z_0 as a function of s for various dielectric materials.

- The various line parameters are:

$$R' = 0 \quad (\sigma_c = \infty) \tag{66}$$

$$G' = 0 \quad (\sigma = 0) \tag{67}$$

$$C' = \frac{\sqrt{\epsilon_{\text{eff}}}}{Z_0 c} \tag{68}$$

$$L' = Z_0^2 C' \tag{69}$$

$$\alpha = 0 \quad (R' = G' = 0) \tag{70}$$

$$\beta = \frac{\omega}{c}\sqrt{\epsilon_{\text{eff}}} \tag{71}$$

- The expressions given allow us to determine Z_0 if we are given the dimensions and material of the microstrip transmission line (i.e, ϵ_r, h and w).

- But, to design a microstrip for a desired Z_0 is more difficult. So, a family of curves are generated so that s can be estimated from a given Z_0. We get two expressions for different Z_0 regions and assume ϵ_r is given (typical values range from 2 to 15),

- For $Z_0 \leq (44 - 2\epsilon_r)\ \Omega$,

$$s = \frac{w}{h} \tag{72}$$

$$s = \frac{2}{\pi}\left[(q-1) - \ln(2q-1) + \frac{\epsilon_r - 1}{2\epsilon_r}\left[\ln(q-1) + 0.29 - \frac{0.52}{\epsilon_r}\right]\right] \tag{73}$$

with

$$q = \frac{60\pi^2}{Z_0\sqrt{\epsilon_r}} \tag{74}$$

- For $Z_0 \geq (44 - 2\epsilon_r)\ \Omega$,

$$s = \frac{8e^p}{e^{2p} - 2} \tag{75}$$

with,

$$p = \sqrt{\frac{\epsilon_r + 1}{2}} \frac{Z_0}{60} + \left[\frac{\epsilon_r - 1}{\epsilon_r + 1}\right] \left[0.23 + \frac{0.12}{\epsilon_r}\right] \tag{76}$$

Microstrip Line Example: What is the width of the copper strip for a microstrip line with $Z_0 = 50\ \Omega$, and 0.5 mm thick sapphire substrate with $\epsilon_r = 9$?

Solution: $44 - 2\epsilon_r = 44 - 2 \times 9 = 44 - 18 = 26\ \Omega$ (note, typo in book), so $Z_0 = 50$ is greater than this so we use,

$$p = \sqrt{\frac{\epsilon_r + 1}{2}} \frac{Z_0}{60} + \left[\frac{\epsilon_r - 1}{\epsilon_r + 1}\right]\left[0.23 + \frac{0.12}{\epsilon_r}\right] \tag{77}$$

when we plug in the numbers we get $p = 2.06$. We then get

$$s = \frac{w}{h} = \frac{8e^p}{e^{2p} - 2} = 1.056 \tag{78}$$

So, $w = sh = 1.056 \times 0.5$ mm $= 0.53$ mm. We can check by using $s = 1.056$ and $\epsilon_r = 9$ and plug into

$$Z_0 = \frac{60}{\sqrt{\epsilon_{\text{eff}}}} \ln\left[\frac{6 + (2\pi - 6)e^{-1}}{s} + \sqrt{1 + \frac{4}{s^2}}\right] \tag{79}$$

along with the other necessary equations and this gives $Z_0 = 49.93\ \Omega$.

2.6. Lossless transmission line

The most general case is a bit too complicated, so let's make some (very good) approximations:

- Use a conductor (wires) with low resistance; this minimizes ohmic losses so that $R' \ll \omega L'$.
- Use a very good dielectric between the conductors, so that $G' \ll \omega C'$
- $R' \approx 0$ and $G^{'} \approx 0$ results in (lossless case $\alpha = 0$):

$$\gamma = \alpha + j\beta = 0 + j\omega\sqrt{L'C'} \tag{80}$$

- And $\beta = \omega\sqrt{L'C'}$

- This significantly simplifies the expression for characteristic impedance

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \sqrt{\frac{L'}{C'}}\ (\Omega) \tag{81}$$

$$Z_0 = \sqrt{\frac{L'}{C'}}\ (\Omega) \tag{82}$$

- What do you notice about the simplified equation relative to the original?

- Other quantities can now also be expressed in a simple form

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{L'C'}}, \quad u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}} \tag{83}$$

- Using the relationship, $u_p = \frac{1}{\sqrt{\mu\epsilon}}$,

$$\beta = \omega\sqrt{\mu\epsilon}\ (\text{rad/m}) \tag{84}$$

- Typical materials for transmission lines will have permeability $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m, while permittivity is given via *relative permittivity* $\epsilon_r = \epsilon/\epsilon_0$. Free space permittivity $\epsilon_0 = 8.854 \times 10^{-12}$ F/m.

- Some simple manipulations leads to

$$u_p = \frac{c}{\sqrt{\epsilon_r}} \Rightarrow \lambda = \frac{u_p}{f} = \frac{c}{f}\frac{1}{\sqrt{\epsilon_r}} = \frac{\lambda_0}{\sqrt{\epsilon_r}} \tag{85}$$

One more property of transmission lines: dispersive or not? If the phase velocity is independent of frequency ⇒ medium is **nondispersive**. Lossless TEM lines are of this type. Why do we care? It gets to the signal integrity issues and how faithfully is the shape of the signal preserved (fig. 2).

Summary of several cases of transmission lines is given in table 2.2.

• Voltage reflection coefficient

After all this, we still don't know the voltage! Actually, to solve eqs. 87 we need a full circuit, as presented in fig. 10.

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \tag{86}$$

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \tag{87}$$

Remember that for lossless lines $\gamma = j\beta$. Set up coordinate system so that $z = 0$ at the load end and $z = -l$ at the generator end.

- We know that at the load

$$Z_L = \tilde{V}_L / \tilde{I}_L \tag{88}$$

  This has nothing to do with the waves!

- Now we make a connection with the wave picture:

$$\tilde{V}_L = \tilde{V}(z=0) = V_0^+ + V_0^-, \ \tilde{I}_L = \tilde{I}(z=0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} \tag{89}$$

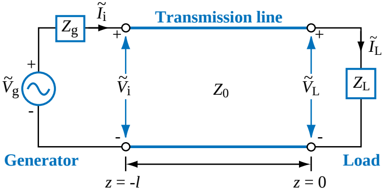


Figure 10: Transmission line connected to a general load and generator.

- Plug this into eq. 88

$$\Rightarrow Z_L = \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right) Z_0 \Rightarrow V_0^- = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) V_0^+ \qquad (90)$$

- Very interesting result: the ratio of incident and reflected wave at the load depends only on the load impedance Z_L and characteristic impedance Z_0!

- This we call **voltage reflection coefficient**

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} \tag{91}$$

- Similarly, for current waves

$$\frac{I_0^-}{I_0^+} = -\frac{V_0^-}{V_0^+} = -\Gamma \tag{92}$$

- Note that Γ is a **complex** number! Z_0 may be real (for lossless lines), but Z_L is generally complex.

$$\Rightarrow \Gamma = |\Gamma| e^{j\Theta_r} \tag{93}$$

- For passive loads $|\Gamma| \leq 1$
- The load is considered matched if $Z_L = Z_0 \Rightarrow$ no reflection at the load $\Rightarrow V_0^- = 0$

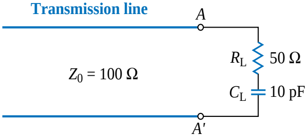


Figure 11: Transmission line for example 2-3.

- Open-circuit (O-C) load $\Rightarrow \Gamma = 1, V_0^- = V_0^+$.
- Short-circuit (S-C) load $\Rightarrow \Gamma = -1, V_0^- = -V_0^+$

- **Standing waves**

We are making progress, but we still don't have a full solution! We've made it this far:

$$\tilde{V}(z) = V_0^+(e^{-j\beta z} + \Gamma e^{j\beta z}), \tilde{I}(z) = \frac{V_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z}) \qquad (94)$$

(What is the unknown?)

- Let's do a little bit more examination by looking at the magnitude of $\tilde{V}(z)$.

- After some manipulation

$$|\tilde{V}(z)| = |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta z + \Theta_r)\right]^{1/2} \qquad (95)$$

  and similarly for $|\tilde{I}(z)|$.

- Fig. 12 shows these magnitudes vs. position z, given the circuit parameters.

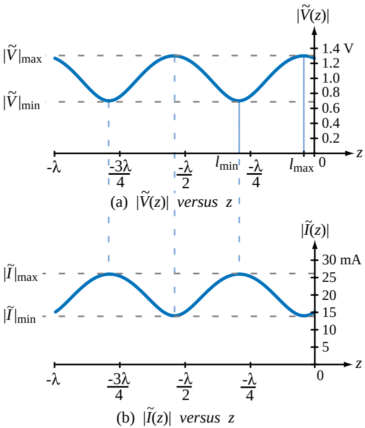


Figure 12: Standing voltage ($\tilde{V}$) and current ($\tilde{I}$) waves. $Z_0 = 50\Omega$, $\Gamma = 0.3e^{j30^\circ}$, $|V_0^+| = 1$

- If we substitute in a position $-d$ on our transmission line we have,

$$|\tilde{V}(z)| = |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma| \cos(-2\beta d + \Theta_r)\right]^{1/2} \quad (96)$$

$$|\tilde{V}(z)| = |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta d - \Theta_r)\right]^{1/2} \quad (97)$$

Some observations:

- The magnitudes show a sinusoidal pattern, which is caused by **interference** of two waves,
- This pattern is called standing wave,
- The maximum of the standing wave pattern happens when incident and reflected waves are in phase, i.e. the argument of the cosine term, $2\beta d - \Theta_r = 2n\pi$. In this case, magnitude of total voltage is $(1 + |\Gamma|)|V_0^+|$.

- Conversely, if the traveling waves are **out of phase**, we have a minimum. This happens at $2\beta d - \Theta_r = (2n+1)\pi$, and magnitude is $(1 - |\Gamma|)|V_0^+|$.

- Standing wave pattern repeats every $\lambda/2$, where λ is associated with the traveling waves.

- Note that fig. 12 is vs. **coordinate** z, i.e. there is **no time dependence**, which is OK since we are looking at magnitudes (or amplitudes) only.

- What happens if we fix the position and look at time? Voltage has a $\cos \omega t$ variation.

- Interestingly, current and voltage are in opposition; when voltage peaks, current has a minimum and vice versa. This is a consequence of a minus sign in eq. 94.

There are three other special cases: matched load, O-C and S-C, which are shown in fig. 13.

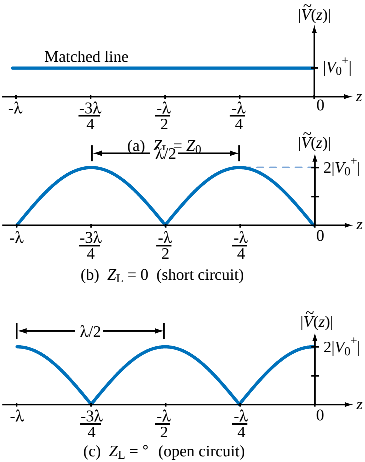


Figure 13: Standing waves for matched load, S-C and O-C.

- The matched load $\Rightarrow |\Gamma| = 0 \Rightarrow |\tilde{V}(z)| = |V_0^+|$, i.e. without any reflected waves there can be no interference $\Rightarrow$ no standing waves.

- S-C and O-C cases have $|\Gamma| = 1$, or $\Gamma = -1$ for S-C and $\Gamma = 1$ for O-C.

- S-C and O-C have the same maximum value: $2|V_0^+|$, and minimum value of zero. Their patterns are shifted by $\lambda/4$.

- The first voltage minimum is at $z = 0$ for S-C, while O-C has first maximum at $z = 0$; why?

What about a general expression for the position of the first maximum (or minimum)? We've already seen that when

$$2\beta d_{max} - \theta_r = 2n\pi \Rightarrow |\tilde{V}|_{max} = |V_0^+|[1 + |\Gamma|] \tag{98}$$

with n=0 or a positive integer. So, what's d_{max}?

$$d_{max} = \frac{\theta_r + 2n\pi}{2\beta} = \frac{\theta_r \lambda}{4\pi} + \frac{n\lambda}{2} \tag{99}$$

where $n = 1, 2, \ldots$ if $\theta_r < 0$, and $n = 0, 1, 2 \ldots$ if $\theta_r \geq 0$.

- θ_r is bounded by $-\pi$ and π,
- When $\theta_r \geq 0$ the first $d_{max} = \theta_r \lambda / 4\pi$, otherwise it occurs at $d_{max} = (\theta_r \lambda / 4\pi) + \lambda/2$.
- Maximum of voltage standing wave is also where current standing wave has a minimum!

- Derivation of positions for minima is analogous and yields

$$|\tilde{V}|_{min} = |V_0^+|[1 - |\Gamma|], \text{ for } (2\beta d_{min} - \theta_r) = (2n+1)\pi \quad (100)$$

and the first minimum occurs for $n = 0$.

- Spacing between d_{max} and d_{min} is $\lambda/4 \Rightarrow$ no need to calculate minima separately:

$$\begin{aligned} d_{min} &= d_{max} + \lambda/4, \text{ if } d_{max} < \lambda/4 \\ d_{min} &= d_{max} - \lambda/4, \text{ if } d_{max} \geq \lambda/4 \end{aligned} \quad (101)$$

- Finally, the ratio $|\tilde{V}_{max}|/|\tilde{V}_{min}|$ is the voltage standing wave ratio S, aka SWR or VSWR:

$$S = \frac{|\tilde{V}_{max}|}{|\tilde{V}_{min}|} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad (102)$$

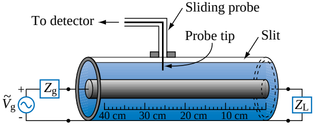


Figure 14: Slotted coaxial line.

2.7. Input Impedance

We've learned a lot without actually solving our original equations

- We understand that the voltage and current magnitudes are oscillatory with position on the line and are out of phase with each other.

- Since impedance is the ratio of voltage to current, we can talk about the input impedance which is,

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} \tag{103}$$

- Substitute the solution we found for $\tilde{V}(z)$ and $\tilde{I}(z)$

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{V_0^+ \left[e^{j\beta d} + \Gamma e^{-j\beta d}\right]}{V_0^+ \left[e^{j\beta d} - \Gamma e^{-j\beta d}\right]} Z_0 = Z_0 \frac{1 + \Gamma e^{-j2\beta d}}{1 - \Gamma e^{-j2\beta d}} \tag{104}$$

- Where we can define, $\Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j(\theta_r - 2\beta d)}$. Note the book also uses Γ_l which is just when $d = l$ at the end of the line.

- Do not confuse Z with Z_0! The former is ratio of total voltage and current, while the latter is the ratio of the individual traveling wave components! Remember that any point total voltage (or current) is a sum of the traveling components!

- Note that, if $d = l$ we are at a position at the input (at the source) of the transmission line. We can define that as the input impedance,

$$Z_{in}(l) = Z_0 \frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-j2\beta l}} = Z_0 \frac{1 + \Gamma_l}{1 - \Gamma_l} \tag{105}$$

can be calculated for a known load impedance and properties of the transmission line.

$$\Gamma_l = \Gamma e^{-j2\beta l} = |\Gamma| e^{j(\theta_r - 2\beta l)} \tag{106}$$

- Since we know how to calculate Γ,

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{107}$$

We can use what we found earlier

$$Z(d) = Z_0 \frac{\left[e^{j\beta d} + \Gamma e^{-j\beta d}\right]}{\left[e^{j\beta d} - \Gamma e^{-j\beta d}\right]} \tag{108}$$

and we can substitute,

$$e^{j\beta l} = \cos\beta l + j\sin\beta l \tag{109}$$

$$e^{-j\beta l} = \cos\beta l - j\sin\beta l \tag{110}$$

With some manipulations we can get to,

$$Z_{in} = Z_0 \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} \tag{111}$$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \tag{112}$$

We can also use the normalized load impedance,

$$z_L = \frac{Z_L}{Z_0} \tag{113}$$

$$Z_{in} = Z_0 \frac{z_L + j\tan\beta l}{1 + jz_L \tan\beta l} \tag{114}$$

We can finally solve for the voltage. This is where the "circuit" and "wave" approaches meet.

- We have basically solved for the impedance of the transmission line and if we are at the source with the load Z_L we have Z_{in} (as in fig. 15) so we can replace everything to the right of the source with Z_{in}.

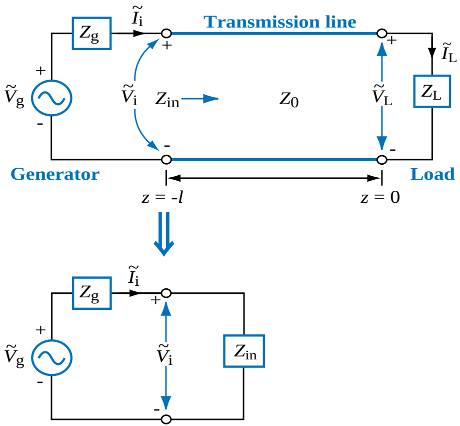


Figure 15: Input impedance of transmission line + load.

- Once the transmission line and load are replaced by Z_{in}, finding voltage at the input of transmission line is easy:

$$\tilde{I}_i = \frac{\tilde{V}_g}{Z_g + Z_{in}} \tag{115}$$

The voltage across Z_{in} is then,

$$\tilde{V}_i = \tilde{I}_i Z_{in} = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} \tag{116}$$

and this is the situation from the standpoint of the source ("circuit" picture)

- From the transmission line, we solved for the voltage as the sum of the two waves:

$$\tilde{V}_i = \tilde{V}(-l) = V_0^+[e^{j\beta l} + \Gamma e^{-j\beta l}] \tag{117}$$

- Clearly, the two results must give one and the same voltage so we can equate the two,

$$\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} = V_0^+[e^{j\beta l} + \Gamma e^{-j\beta l}] \tag{118}$$

And, solve for the unknown V_0^+,

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) \tag{119}$$

$\Rightarrow$ and we have our solution!

2.8. Special Cases

Here we'll have a closer look at some special transmission lines: Short-Circuited, Open Circuited, Matched Lines, $\lambda/2$ lines and $\lambda/4$ lines.

- **Short-circuited line**

Now that we know how to calculate Z_{in}, let's have a look at the short-circuited case. That corresponds to $Z_L = 0$ so we can easily calculate Γ:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{-Z_0}{Z_0} = -1 \tag{120}$$

The standing wave ratio is given by,

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \infty \tag{121}$$

- Voltage and current variation with z can easily be obtained using

$$\tilde{V}(d) = V_0^+[e^{j\beta d} + \Gamma e^{-j\beta d}] \tag{122}$$

$$\tilde{I}(d) = \frac{V_0^+}{Z_0}[e^{j\beta d} - \Gamma e^{-j\beta d}] \tag{123}$$

$$\tilde{V}_{sc}(d) = V_0^+[e^{j\beta d} - e^{-j\beta d}] = 2jV_0^+ \sin\beta d \tag{124}$$

$$\tilde{I}_{sc}(d) = \frac{V_0^+}{Z_0}[e^{j\beta d} + e^{-j\beta d}] = \frac{2V_0^+}{Z_0}\cos\beta d \tag{125}$$

  and is shown in fig. 16. Note the locations of maxima and minima.

- Voltage is zero at the load ($d = 0$) as it should be for short circuit.

- What about input impedance? For a line of length l,

$$Z_{in}^{sc} = \frac{\tilde{V}_{sc}(l)}{\tilde{I}_{sc}(l)} = jZ_0 \tan\beta l \tag{126}$$

which is shown if fig. 16.

- Very interesting: Short circuit transmission line has an impedance that is purely reactive (no real part), when (using now length of line l), $\tan\beta l > 0$ line appears inductive; opposite case $\Rightarrow$ capacitive.

- Say we set up an inductor with the same inductance ($\tan\beta l > 0$), i.e.
  $j\omega L_{eq} = jZ_0 \tan\beta l$

$$\rightarrow L_{eq} = \frac{Z_0 \tan\beta l}{\omega} \text{ (H)} \tag{127}$$

- If Z_0 is fixed, then the only variable is length l, and the minimum length to obtain inductance L_{eq} is

$$l = \frac{1}{\beta}\tan^{-1}\left(\frac{\omega L_{eq}}{Z_0}\right) \text{ (m)} \tag{128}$$

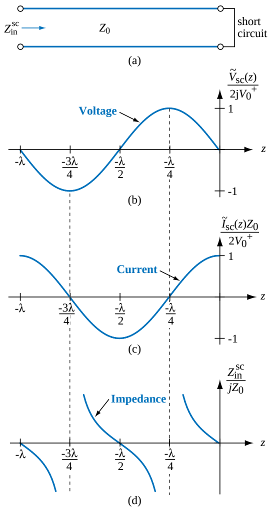


Figure 16: Input impedance of a S-C transmission line.

- What if $\tan\beta l < 0$? Then, the short-circuit transmission line appears capacitive, and the equivalent capacitance is

$$\frac{1}{j\omega C_{eq}} = jZ_0 \tan\beta l \text{ (F)} \tag{129}$$

$$C_{eq} = -\frac{1}{Z_0\omega\tan\beta l} \text{ (F)} \tag{130}$$

- Since βl is a positive number, the first place we can get a negative number (on the tan curve) is when $\beta l = \pi/2$ and continues negative until $\beta l = \pi$. To find the minimum l for capacitive short-circuit transmission line,

$$\frac{-1}{C_{eq}Z_0\omega} = \tan\beta l \tag{131}$$

$$\beta l = -\tan^{-1}\left(\frac{1}{\omega C_{eq} Z_0}\right) \tag{132}$$

since $\tan^{-1}(-\theta) = -\tan^{-1}(\theta)$. But, this would give us a negative number. Remember though that the tan curve repeats itself every π so the minimum value for l would be the above plus π:

$$\beta l = \pi - \tan^{-1}\left(\frac{1}{\omega C_{eq} Z_0}\right) \text{ (m)} \tag{133}$$

$$l = \frac{1}{\beta}\left[\pi - \tan^{-1}\left(\frac{1}{\omega C_{eq} Z_0}\right)\right] \text{ (m)} \tag{134}$$

- This means that depending on our choice for l the short circuit line can be a substitute for capacitors or inductors.

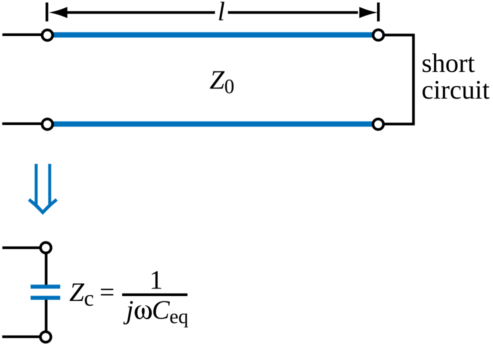


Figure 17: Short-circuit transmission line as equivalent capacitor.

- **Open-circuited line**

Situation is analogous to the short-circuit case. For $Z_L = \infty$, we have $\Gamma = 1$ and voltage and current,

$$\tilde{V}(z) = V_0^+[e^{-j\beta z} + \Gamma e^{j\beta z}] \tag{135}$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0}[e^{-j\beta z} - \Gamma e^{j\beta z}] \tag{136}$$

$$\tilde{V}_{oc}(d) = V_0^+[e^{j\beta d} + e^{-j\beta d}] = 2V_0^+ \cos\beta d \tag{137}$$

$$\tilde{I}_{oc}(d) = \frac{V_0^+}{Z_0}[e^{j\beta d} - e^{-j\beta d}] = \frac{2jV_0^+}{Z_0}\sin\beta d \tag{138}$$

Input impedance is again the ratio (when $d = l$),

$$Z_{in}^{oc} = \frac{\tilde{V}_{oc}(l)}{\tilde{I}_{oc}(l)} = -jZ_0 \cot\beta l \tag{139}$$

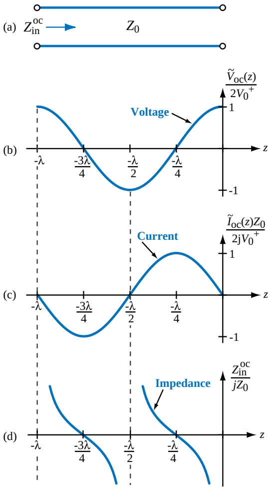


Figure 18: Input impedance of a O-C transmission line.

• Application of short-circuit and open-circuit measurements

If we measure a transmission line (we can use a network analyzer) that is short circuited and get Z_{sc} and then measure again open circuit we get Z_{oc}, we can combine to determine the characteristic impedance Z_0 and β.

- The product of impedances is,

$$Z_{in}^{oc} Z_{in}^{sc} = -jZ_0 \cot \beta l jZ_0 \tan \beta l = Z_0^2 \tag{140}$$

so,

$$Z_0 = \sqrt{Z_{in}^{oc} Z_{in}^{sc}} \tag{141}$$

Similarly, take the ratio,

$$\frac{Z_{in}^{sc}}{Z_{in}^{oc}} = \frac{jZ_0 \tan \beta l}{-jZ_0 \cot \beta l} = -\tan^2 \beta l \tag{142}$$

$$\tan \beta l = \sqrt{-\frac{Z_{in}^{sc}}{Z_{in}^{oc}}} \tag{143}$$

- **One-half wavelength lines**

The main point about this line is that multiples of $\lambda/2$ lengths do nothing in terms of input impedance!

- If $l = n\lambda/2$ then

$$\tan\beta l = \tan[(2\pi/\lambda)(n\lambda/2)] = \tan n\pi = 0 \tag{144}$$

- Recall, the input impedance,

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \tag{145}$$

so,

$$\Rightarrow Z_{in} = Z_L \quad \text{for } l = n\lambda/2 \tag{146}$$

- $\Rightarrow$ Generator connected to Z_L via a transmission line that is multiple of $\lambda/2$ long induces in the load same currents and voltages as if the transmission line were not there.

- Remember that this is valid at **only one frequency**!.

- **Quarter wave transformer**

When transmission line length $l = \lambda/4$ (or $= \lambda/4 + n\lambda/2$), we have another interesting case:

- $\beta l = (2\pi/\lambda) \times (\lambda/4) = \pi/2$ so $\tan \beta l \to \infty$
- Recall, the input impedance,

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \tag{147}$$

- so that

$$Z_{in} = \frac{Z_0^2}{Z_L}, \quad \text{for } l = \lambda/4 + n\lambda/2 \tag{148}$$

- How is this useful?

Illustrate usefulness of the quarter-wave transformer with an example in fig. 19.

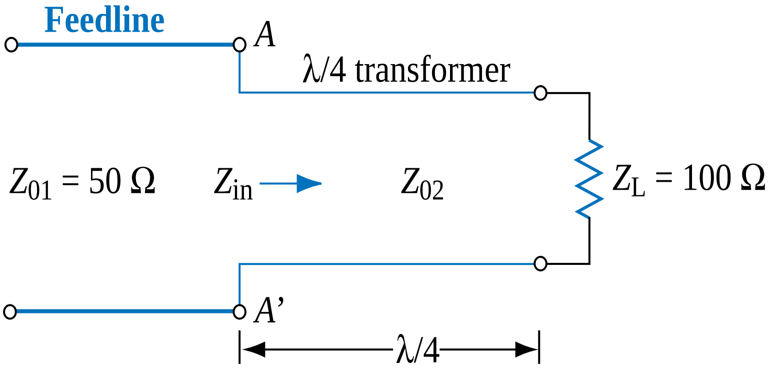


Figure 19: Circuit for example 2-10.

- **Matched line**

In this case $Z_L = Z_0$

- $Z_{in} = Z_0$
- $\Gamma = 0$
- All incident power at the input is delivered to the load, independent of the line length

2.9. Power flow

So far we've only analyzed transmission lines in terms of voltages and currents. There is an alternative point of view, based on **power flow** on the transmission lines that can be quite useful.

- Remember the expressions for voltage and current on transmission line:

$$\tilde{V}(z) = V_0^+ \left(e^{-j\beta z} + \Gamma e^{j\beta z} \right) \tag{149}$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} [e^{-j\beta z} - \Gamma e^{j\beta z}] \tag{150}$$

- We will substitute in $z = -d$ as we have done before,

$$\tilde{V}(d) = V_0^+ \left(e^{j\beta d} + \Gamma e^{-j\beta d} \right) \tag{151}$$

$$\tilde{I}(d) = \frac{V_0^+}{Z_0} [e^{j\beta d} - \Gamma e^{-j\beta d}] \tag{152}$$

- At the load $z = 0$ and things simplify and we can write in terms of incident and reflected waves: $\tilde{V}^i = V_0^+$, $\tilde{V}^r = \Gamma V_0^+$, $\tilde{I}^i = V_0^+/Z_0$ and $\tilde{I}^r = -\Gamma V_0^+/Z_0$.
- How do we find power if we know voltage and current? Two ways to get the answer: in time or phasor domain.
- Instantaneous incident power at the load ($d = 0$) is easy to find:

$$P^i(0,t) = v^i(0,t) \cdot i^i(0,t) = \frac{|V_0^+|^2}{Z_0} \cos^2(\omega t + \phi^+) \text{ (W)} \quad (153)$$

- What about the reflected power at the load?

$$P^r(0,t) = v^r(0,t) \cdot i^r(0,t) = -|\Gamma|^2 \frac{|V_0^+|^2}{Z_0} \cos^2(\omega t + \phi^+ + \theta_r) \text{ (W)} \quad (154)$$

- Note, book includes more general equations for power at location $z = -d$ but we can obtain the same power results by setting $d = 0$.

- More interesting than the instantaneous power is the time-average power, which can be obtained from

$$P_{av} = \frac{1}{T}\int_0^T P(t)dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} P(t)dt \quad (155)$$

- We can use the identity,

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad (156)$$

And, remember that the integral of $\cos\omega t$ over a period is 0, that is,

$$\int_0^T \cos\omega t dt = 0 \quad (157)$$

- Expanding the incident power out,

$$P^i(0,t) = \frac{|V_0^+|^2}{Z_0}(1/2 + 1/2\cos(2\omega t + 2\phi^+))\ (\mathrm{W}) \quad (158)$$

so we have

$$P_{av}^i = \frac{1}{T}\int_0^T P^i(t)dt = \frac{|V_0^+|^2}{2Z_0} \tag{159}$$

- The reflected power is found similarly, so we have the incident and reflected average power (in Watts),

$$P_{av}^i = \frac{|V_0^+|^2}{2Z_0} \text{ (W)} \tag{160}$$

$$P_{av}^r = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0} = -|\Gamma|^2 P_{av}^i \tag{161}$$

$$-\frac{P_{av}^r}{P_{av}^i} = |\Gamma|^2 \tag{162}$$

- This is an important result: the ratio of the reflected and incident powers at the load give the reflection coefficient magnitude squared, or that the reflected power at the load is equal to the incident power reduced by $|\Gamma|^2$ term.

- The net average power delivered to the load is a sum of the incident and reflected powers:

$$P_{av} = P_{av}^i + P_{av}^r = \frac{|V_0^+|^2}{2Z_0}\left[1 - |\Gamma|^2\right] \text{ (W)} \tag{163}$$

- Note, that we could have done the same thing (even easier) using a phasor representation.

- The starting point for that is the average power relationship which is,

$$P_{av} = \frac{1}{2}\Re\left[\tilde{V} \cdot \tilde{I}^*\right] \tag{164}$$

- But, starting with that we still end up with,

$$P_{av} = \frac{|V_0^+|^2}{2Z_0}\left[1 - |\Gamma|^2\right] \text{ (W)} \tag{165}$$

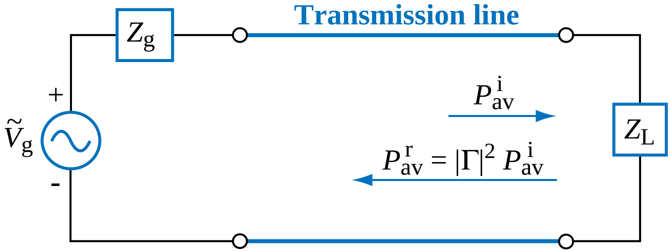


Figure 20: Power flows on a transmission line.

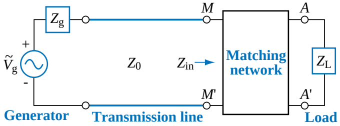


Figure 21: Illustration of matching to transmission line.

2.10. Smith Chart

We won't cover Smith Charts this term.

2.11. Impedance matching

Transmission lines are not just a nuisance that has to be taken into account. We can use them for transforming impedance. Why? Typically, to ensure the best power transfer.

- See Fig. 21 for circuit setup.

- Best case: have a load such that $Z_L = Z_0$. Usually not possible.
- What else? Place **impedance matching** network between the load and transmission line.
- Work with admittances (why?).
- The load admittance is Y_L, the line admittance Y_0 and the shunt admittance is Y_s.
- A the point MM', the admittance of the line to the right is Y_d (includes the length d of line and the load).
- So, the input impedance at a point just to the left of MM' is the sum,

$$Y_{in} = Y_d + Y_s \tag{166}$$

- Generally, Y_d is complex but Y_s will be purely imaginary because we will attach something purely reactive like a capacitor or inductor.

- So,

$$Y_{in} = (G_d + jB_d) + jB_s = G_d + j(B_d + B_s) \tag{167}$$

- Or, the normalized form,

$$y_{in} = g_d + j(b_d + b_s) \tag{168}$$

- To have the line matched, we want $z_{in} = 1$ (because $\Gamma = 0$ for matched line so we have $Z_{in} = Z_0$) and therefore $y_{in} = 1$. So that requires, $g_d = 1$ and $b_d + b_s = 0$. So, we have two conditions,

$$g_d = 1, b_d = -b_s \tag{169}$$

- We have two conditions to satisfy so we have two degrees of freedom. We will use the parameter d (length of line from the load where we attach shunt element) to satisfy the $g_d = 1$ condition. We will use an appropriate capacitor or inductor to satisfy the $b_d = -b_s$ condition.

- To see how to get $g_d = 1$, we have to do some more manipulations.

$$\Gamma = \frac{1 - y_L}{1 + y_L} \tag{170}$$

and,

$$y_d = \frac{1 - \Gamma e^{-j2\beta d}}{1 + \Gamma e^{-j2\beta d}} = \frac{1 - |\Gamma| e^{j(\theta_r - 2\beta d)}}{1 + |\Gamma| e^{j(\theta_r - 2\beta d)}} \tag{171}$$

- We can use Eulers and do some further manipulations to get the real and imaginary parts,

$$g_d = \frac{1 - |\Gamma|^2}{1 + |\Gamma|^2 + 2|\Gamma| \cos(\theta_r - 2\beta d)} \tag{172}$$

and,

$$b_d = \frac{-2|\Gamma| \sin(\theta_r - 2\beta d)}{1 + |\Gamma|^2 + 2|\Gamma| \cos(\theta_r - 2\beta d)} \tag{173}$$

- We can see from g_d that if we want to make that 1 we have to have,

$$\cos(\theta_r - 2\beta d) = -|\Gamma| \tag{174}$$

- That will give us a denominator,

$$1 + |\Gamma|^2 - 2|\Gamma|^2 = 1 - |\Gamma|^2 \tag{175}$$

- So that, we would have,

$$g_d = \frac{1 - |\Gamma|^2}{1 + |\Gamma|^2 + 2|\Gamma|\cos(\theta_r - 2\beta d)} = \frac{1 - |\Gamma|^2}{1 + |\Gamma|^2 - 2|\Gamma|^2} = 1 \tag{176}$$

- So, for the condition,

$$\cos(\theta_r - 2\beta d) = -|\Gamma| \tag{177}$$

  we have to adjust d since that is all we have control over. So, we select a d to do the above.

- Once we fix d, then we have to calculate what b_d is and then add our shunt element so that $b_s = -b_d$.

- Many different ways to accomplish matching. We could add a capacitor or inductor, or we can use transmission lines in an arrangement called "single stub", illustrated in Fig. 22. (Remember a transmission line can be the same as an inductor or capacitor).

- To satisfy both our two degrees of freedom we have two lengths; d and the length of a "stub" l placed in shunt. Stub is either S-C or O-C.

- The procedure:

  1. Convert Z_L to Y_L
  2. Select a distance d so as to transform the load admittance Y_L into $Y_d = Y_0 + jB$ (i.e. so that $y_L = 1$ looking into $MM^{'}$ before adding the stub).

3. Add shunt stub (O-C or S-C) with $Y_s = -jB$ so that total admittance looking into MM' (with stub) is $Y_{in} = Y_0 + jB - jB = Y_0$, as needed!

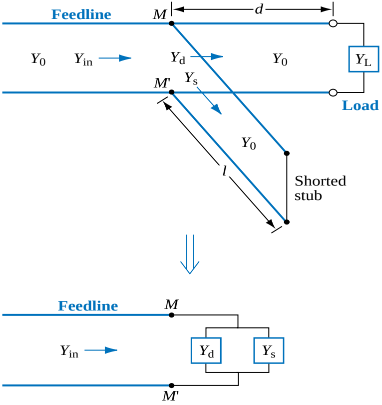


Figure 22: Impedance matching using shorted-stub.

2.12. Transients on Transmission lines

Everything we've done so far was frequency-centric. Also, largely applicable only to narrowband signals. What do we do with wideband applications? For that we need the **transient response** of the transmission line network.

- Start with a single pulse of amplitude V_0 and duration τ.
- Decompose it into two step functions:

$$V(t) = V_1(t) + V_2(t) = V_0 U(t) - V_0 U(t - \tau) \tag{178}$$

  where $U(t)$ is a unit step function. Illustrated in fig. 23.

- Why bother? If we know what happens to a step response we can figure out pulse responses!

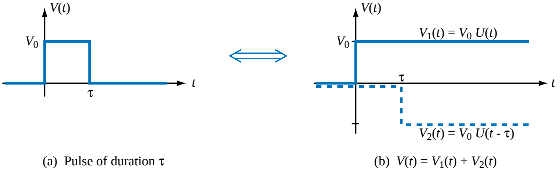
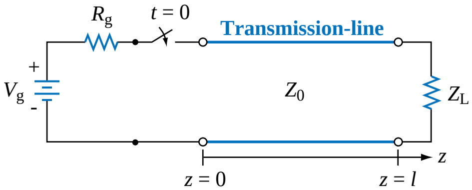


Figure 23: Rectangular pulse as a sum of two step functions.

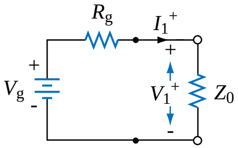
- **Transient response**

For a transient response we need to include switches. A typical setup is presented in Fig. 24: DC source, switch, transmission line and load. Here, $Z_L = R \Rightarrow$ all impedances are real.

- Close the switch at $t = 0$. What's the impedance the source "sees?"
- The voltage V_1 is now an **initial condition**, and is given in Fig. 24.



(a) Transmission-line circuit



(b) Equivalent circuit at $t=0^+$

Figure 24: The transmission line immediately after turn-on.

- How do we get the current?

$$I_1^+ = \frac{V_g}{R_g + Z_0} \tag{179}$$

$$V_1^+ = \frac{V_g Z_0}{R_g + Z_0} \tag{180}$$

- I_1^+, V_1^+ are waves that start traveling down the transmission line. At what velocity? Note: + sign indicates travel in the positive z direction. Note also that we have switched from the previous convention of $z = 0$ at the load to here, $z = 0$ at the generator (more convenient this way).
- A "snapshot" of the voltage and current along transmission line, at three different times is shown in Fig. 25. Note: $R_g = 4Z_0$ and $Z_L = 2Z_0$.
- At first V_1^+ traverses transmission line, then it "hits" the load at $t = T$ ($T = l/u_p$). What happens? If $Z_L \neq Z_0$ we have a

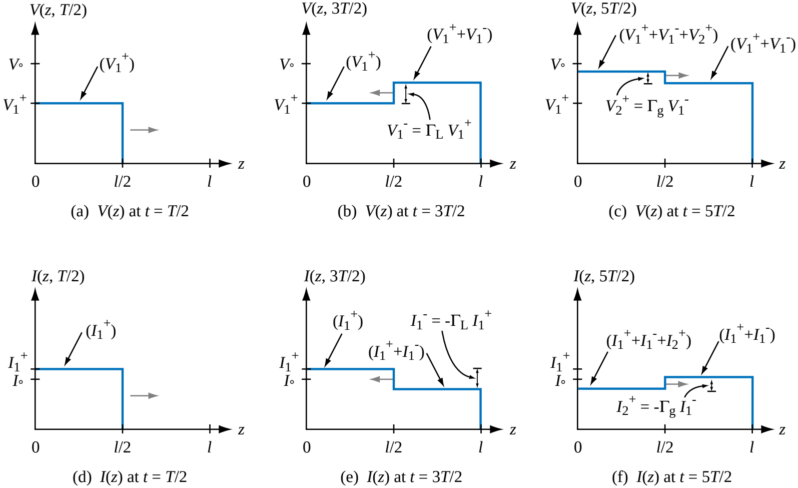


Figure 25: Time evolution of voltage and current on transmission line.

reflection.

$$V_1^- = \Gamma_L V_1^+ \quad \text{and} \quad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1}{3} \qquad (181)$$

- Once we hit the load, we get the negative z traveling wave and we sum the forward and backward traveling waves. V_1^- traverses the transmission line from the load to the generator (source) and when it is half way there — take another snapshot.

- What happens once V_1^- reaches the generator end (at what time?)? If $Z_g \neq Z_0 \Rightarrow$ another reflection!

$$V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+, \quad \text{where} \quad \Gamma_g = \frac{R_g - Z_0}{R_g + Z_0} = 0.6 \quad (182)$$

- V_2^+ starts its travel from the source to the load and the total voltage is a sum of three components. Take snapshot half way

$(t = 5T/2)$.

$$\underbrace{V(z, \frac{5T}{2}) = (1 + \Gamma_L + \Gamma_L \Gamma_g) V_1^+}_{0 \le z < l/2},\ \underbrace{V(z, \frac{5T}{2}) = (1 + \Gamma_L) V_1^+}_{l/2 \le z \le l} \tag{183}$$

- Clearly this will go on for a while ...

- What about the current? The procedure the same, but the reflection coefficients come with a negative sign:

$$I_1^- = -\Gamma_L I_1^+, \quad I_2^+ = -\Gamma_g I_1^- = \Gamma_g \Gamma_L I_1^+ \tag{184}$$

- Is there a limit to this? In eq. 183 we keep adding up components that are products of $\Gamma_L^n \Gamma_g^{n,n-1}$. This infinite series has a limit

$$1 + x + x^2 + \ldots = \frac{1}{1-x} \quad \text{for} \quad |x| < 1. \tag{185}$$

- Net result: we can calculate $\lim_\infty V$

$$V_\infty = V_1^+ \frac{1+\Gamma_L}{1-\Gamma_g\Gamma_L} \tag{186}$$

- After using expressions for V_1^+ (eq. 179), Γ_L (eq. 181), and Γ_g (eq. 182) $\Rightarrow$

$$V_\infty = \frac{V_g Z_L}{R_g + Z_L} \tag{187}$$

What is this? (It's called the *steady-state voltage*)

- For the steady-state current

$$I_\infty = \frac{V_\infty}{Z_L} = \frac{V_g}{R_g + Z_L} \tag{188}$$

• Bounce diagrams

How do we keep track of all these waves bouncing back and forth? We use a graphical method involving *bounce diagrams*, as shown in Fig. 26. Note the coordinates:

- The horizontal axis is the position along transmission line; starts at the generator, at $z = 0$, and ends at the end of transmission line $z = l$.
- The vertical axis is time.
- The reflection coefficients Γ_g and Γ_L at each end are indicated. Note the difference between voltages and currents: currents have negative Γ-s.

So, how do we use the bounce diagram?

- Select a point on a transmission line
- Draw a line parallel to time axis through this point

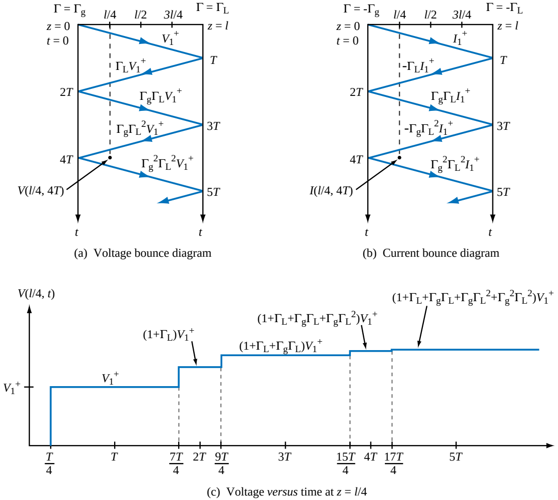
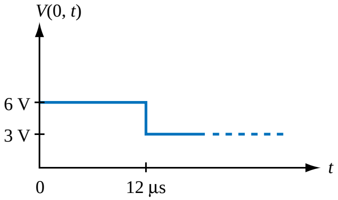


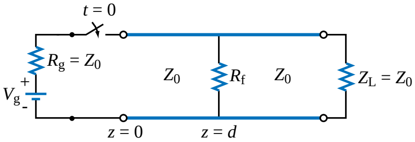
Figure 26: Bounce diagrams and voltage variation at a point.

- Construct a zig-zag curve starting from the generator; note that it takes time $t = T$ to go from one end of the line to another,
- On each section of the curve indicate the product of the reflection coefficients up to that point (whenever the curve "hits" one of the ends, the wave is multiplied by reflection coefficient)
- For your chosen point on transmission line look for its line's intersection with the zig-zag curve; note the times of the intersection
- Between two intersections, the voltage (or current) stays constant
- On a V vs. t diagram indicate the times at which the two lines intersect
- Starting from the first intersection, keep adding up terms (as indicated on the bounce diagram) but remember that between two times voltage is constant.

- For currents, we have to be careful about the signs - best to indicate them on the bounce diagram itself and then just add them up.



(a) Observed voltage at the sending end



(b) The fault at $z = d$ is represented by a fault resistance R_f

Figure 27: Time-domain reflectometer for ex. 2–16.