ECE317
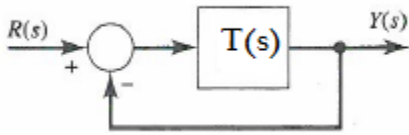## Homework 7

Problem 1: Consider a system with loop gain, $T(s)$, given by:

where $$T(s) = \frac{300(1+s)\left(1+\frac{s}{40}\right)}{s^3}$$

1) Determine whether the system is stable by finding the closed loop poles of the system using the Matlab 'roots' command.

2) We will next determine the gain and phase margins of the system. Use the Matlab 'margin' command to produce a Bode plot of the loop gain.

3) Reading the values from the plot as best as you can determine the frequency at which the magnitude plot is 0dB. This is called the unity gain crossover frequency, ω_C.

4) Determine whether the phase is above or below -180 degrees at this frequency and consequently whether this represents a positive or negative phase margin (Pm).

5) Next read the value from the plot as best as you can determine the frequency at which the phase plot is -180 degree. We will denote this frequency, ω_M.

6) Determine whether the magnitude is above or below 0dB at this frequency and consequently whether this represents a positive or negative gain margin (Gm).

7) From the title line of the Bode plot to determine the values that Matlab gives for ω_C, Pm, ω_M, and Gm.

8) Using the information from the Bode plot alone determine whether the closed loop system is stable or not. Explicitly state the rule(s) which determine this.

9) Is your result of part (8) consistent with the result that would be gleaned by examining the roots obtained in part (1). Explain.

Problem 1 solution:

Matlab script

```
s = tf('s');
T = 300*(1+s)*(1+s/40)/s^3;

cl_sys0 = T/(1+T)

% using 'minreal' will get rid of the
%    equal number of poles and zeros at 0
cl_sys = minreal(T/(1+T))

roots(cl_sys.den{1})

% Alternative method to find pole locations
cl_sys_zpk = zpk(cl_sys);
cl_sys_zpk.p{1}

margin(T)
```

1) Using 'roots' the closed loop poles are found:

      `-3.2517 +17.0425i`

      `-3.2517 -17.0425i`

      `-0.9966 + 0.0000i`

   All are stable poles.

2) See plot below resulting from the 'margin' command.

3) $\omega_C = 18.2$ rad/s

4) Phase at ω_C is above the -180 deg. line so this represents a positive phase margin.

5) $\omega_M = 6.32$ rad/s

6) Magnitude at ω_M is above the 0 dB line so this represents a negative gain margin.

7) $\omega_C = 18.2$ rad/s, $Pm = 21.3$ deg, $\omega_M = 6.32$ rad/s, $Gm = -17.7$ dB

8) $Pm > 0$ ➔ the closed loop system is stable. We do not look at the gain margin (Gm) to determine absolute stability.

9) In part (1) we found the system to be stable, by examining the closed loop system poles, and this is confirmed in part (8) using the loop gain Bode plot stability test.

Bode Diagram
Gm = -17.7 dB (at 6.32 rad/s) , Pm = 21.3 deg (at 18.2 rad/s)
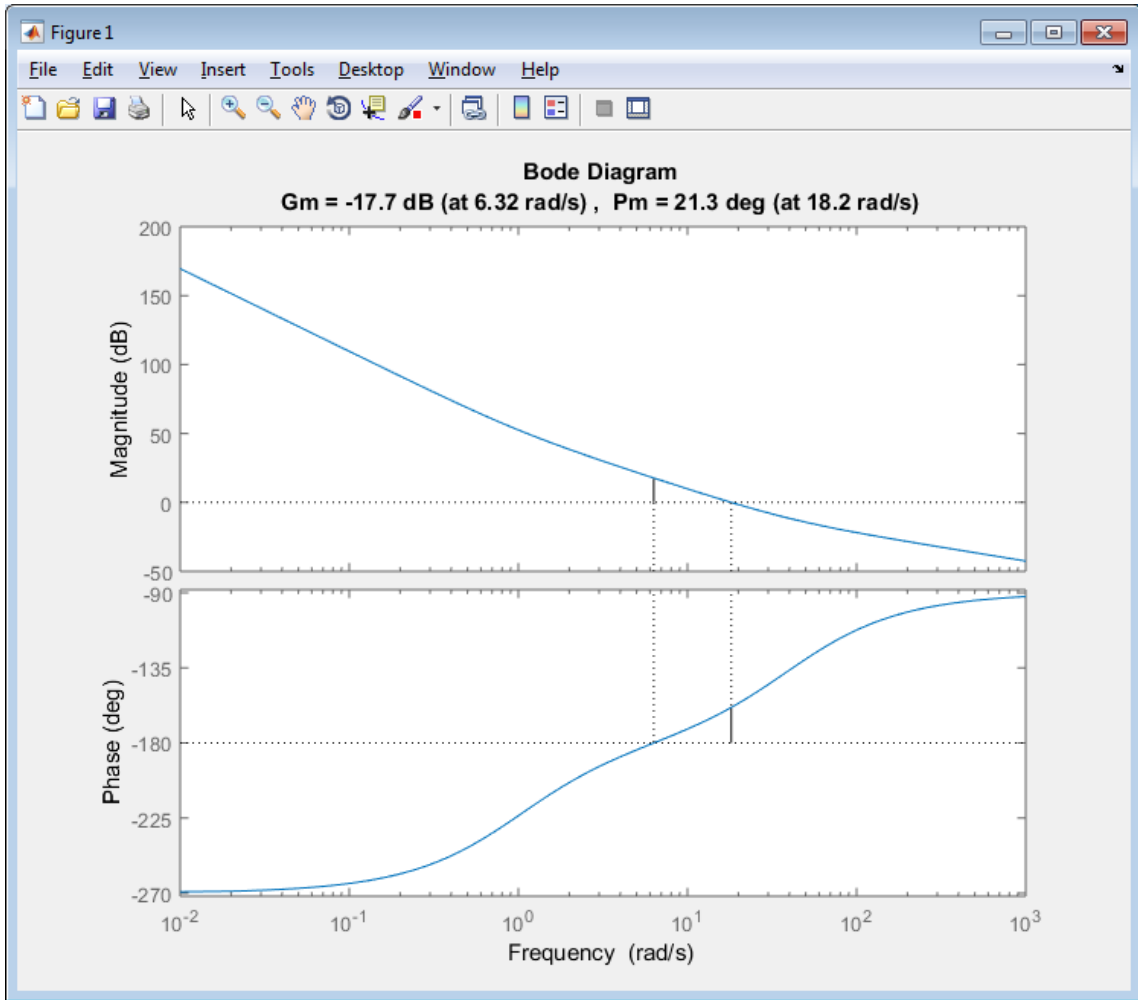
By looking at the curves one can identify a **negative gain margin** and **positive phase margin**

Problem 2:

(Example from class notes)

Sketch the asymptotic Bode plot for the following loop gain.

Annotate the plots completely:
  a) Show the values of all break frequencies for magnitude and phase,
  b) For magnitude plots show:
       i)   gain along all straight line segments, and
       ii)   slopes
  c) For phase plots show:
       i)   phase along zero slope line segments, and
       ii)   the slopes along sloping line segments

$$L(s) = \frac{2500}{s(s+5)(s+50)}$$

Problem 2 Solution:

(solution done in class follows)
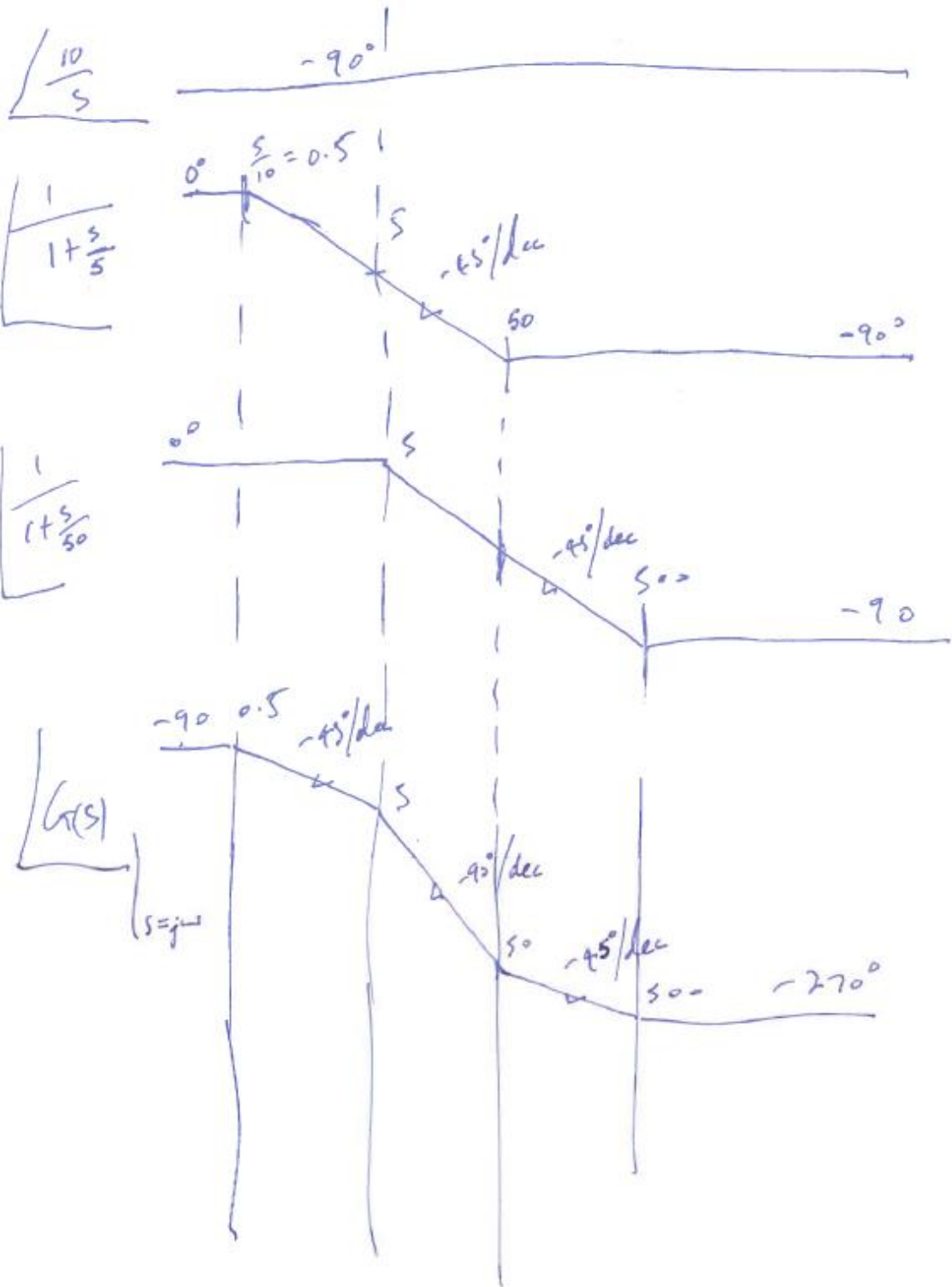
$$L(s) = \frac{2500}{s(s+5)(s+50)} = \underbrace{\left(\frac{2500}{5 \times 50}\right)}_{10} \frac{1}{s\left(1+\frac{s}{5}\right)\left(1+\frac{s}{50}\right)}$$

①

$$G(s) = \frac{10}{s\left(1+\frac{s}{5}\right)\left(1+\frac{s}{50}\right)}$$

$\angle \dfrac{10}{5}$ —90°

$\angle \dfrac{1}{1+\dfrac{s}{5}}$ 0° $\dfrac{s}{10}=0.5$ 5   —45°/dec   50   —90°

$\angle \dfrac{1}{1+\dfrac{s}{50}}$ 0°   5   —45°/dec   500   —90

$\angle G(s)$ —90 0.5 —45°/dec 5 —45°/dec 50 —45°/dec 500 —270°

$s=j\omega$
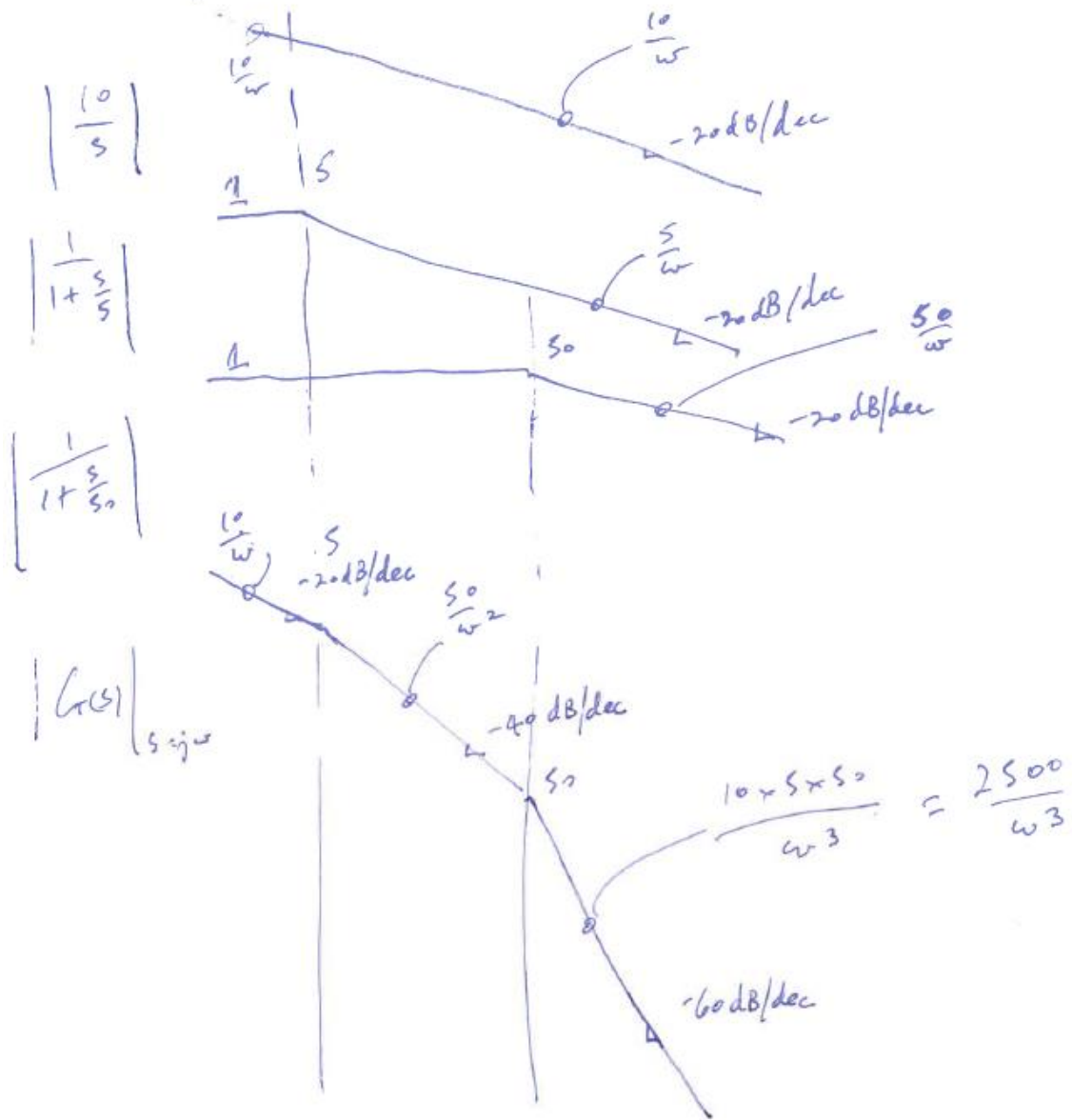
②
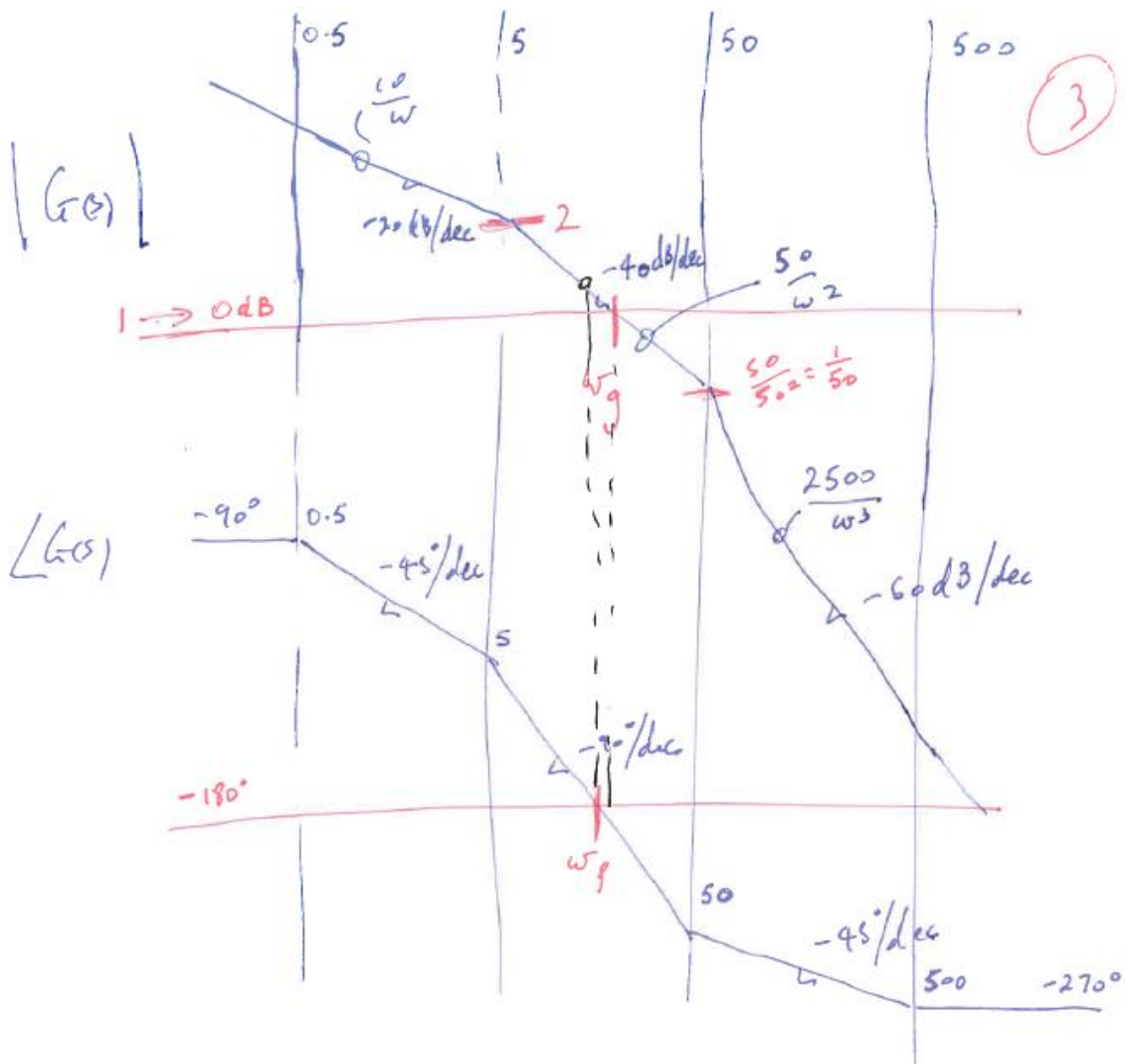
UNITY GAIN CROSSOVER FREQ ω_g

$$\frac{50}{\omega_g^2} = 1 \implies \omega_g = \sqrt{50} = \boxed{7.0711}$$

EXACT PHASE $= -90 - atan\left(\frac{\omega}{5}\right) - atan\left(\frac{\omega}{50}\right)$

$$-90 - atan\left(\frac{\omega_p}{5}\right) - atan\left(\frac{\omega_p}{70}\right) = -180$$

$$\implies 90 - atan\left(\frac{\omega_p}{5}\right) - atan\left(\frac{\omega_p}{50}\right) = 0. \implies \boxed{\omega_p = 15.81}$$

SEE LATTER
PAGES TO SEE
HOW THIS
SOLUTION CAN BE
OBTAINED BY
2 DIFFERENT METHODS

$$\text{phase} \atop @ \, \omega_g = -90 - atan\left(\frac{\omega_g}{5}\right) - atan\frac{\omega_g}{50}$$

$$= -152.7°$$

$$\Rightarrow \boxed{27.2° \quad Pm.}$$

GAIN
MARGIN

$$Gain\left[@ \, \omega_p = 15.91\right] = \frac{50}{\omega^2}\bigg|_{\omega = 15.81}$$

$$= \frac{50}{(15.81)^2}$$

GAIN MARGIN

$$= 0.2 \quad \Rightarrow \quad = 20\log_{10}\left(\frac{1}{0.2}\right) = \boxed{14\,dB}$$

SUMMARY OF RESULTS

MATLAB "MARGIN"

$$\Rightarrow Gm = 14.8\,dB. \quad @ \, 15.8\,rds/se.$$

$$Pm = 31.7° \qquad @ \, 6.22\,rds/s.$$

'ASYMPTOTIC'
ANALYSIS $GM = 14\,dB$ @ 15.81 rds/s
GAVE $Pm = 27.2°$ @ 7.0711 rds/sec.

Using Matlab to find the phase crossover frequency:

In class we solved the problem of finding the -180 degree phase crossover frequency by solving numerically using the Matlab function 'fzero'. The function finds the zero of a nonlinear equation (given an initial guess to the solution).

In Matlab we declare the nonlinear equation using the Matlab 'anonymous function' concept. (A general Matlab *.m file type function may also be used.)

The following two lines is all that is required to solve the problem using Matlab:

```
ph180 = @(wp) 90-atand(wp/5)-atand(wp/50) % this is the anonymous function
fzero(ph180, 10)   % our initial guess is wp = 10.
```

This gives the solution wp = 15.81

Alternatively:

An alternative solution approach uses the following trigonometric identity:

$$\tan^{-1}(A) + \tan^{-1}(B) = \tan^{-1}\left(\frac{A+B}{1-A\cdot B}\right)$$

Starting with the equation

$$-90 - \tan^{-1}\left(\frac{\omega_p}{5}\right) - \tan^{-1}\left(\frac{\omega_p}{50}\right) = 0$$

Using the above identity we have

$$\tan^{-1}\left(\frac{\frac{\omega_p}{5} + \frac{\omega_p}{50}}{1 - \frac{\omega_p^2}{250}}\right) = -90$$

Which reduces to

$$\left(\frac{\frac{\omega_p}{5} + \frac{\omega_p}{50}}{1 - \frac{\omega_p^2}{250}}\right) = \tan(-90) = -\infty$$

$$\rightarrow \qquad 1 - \frac{\omega_p^2}{250} = 0$$

$$\rightarrow \qquad \omega_p = \sqrt{250}$$

$$\rightarrow \quad \omega_p = 15.81 \quad \text{(which is the same result given by Matlab)}$$

Exact results from Matlab:

```
s=tf('s')
sys = 2500/(s*(s+5)*(s+50))
margin(sys)
```

Bode Diagram
Gm = 14.8 dB (at 15.8 rad/s) , Pm = 31.7 deg (at 6.22 rad/s)
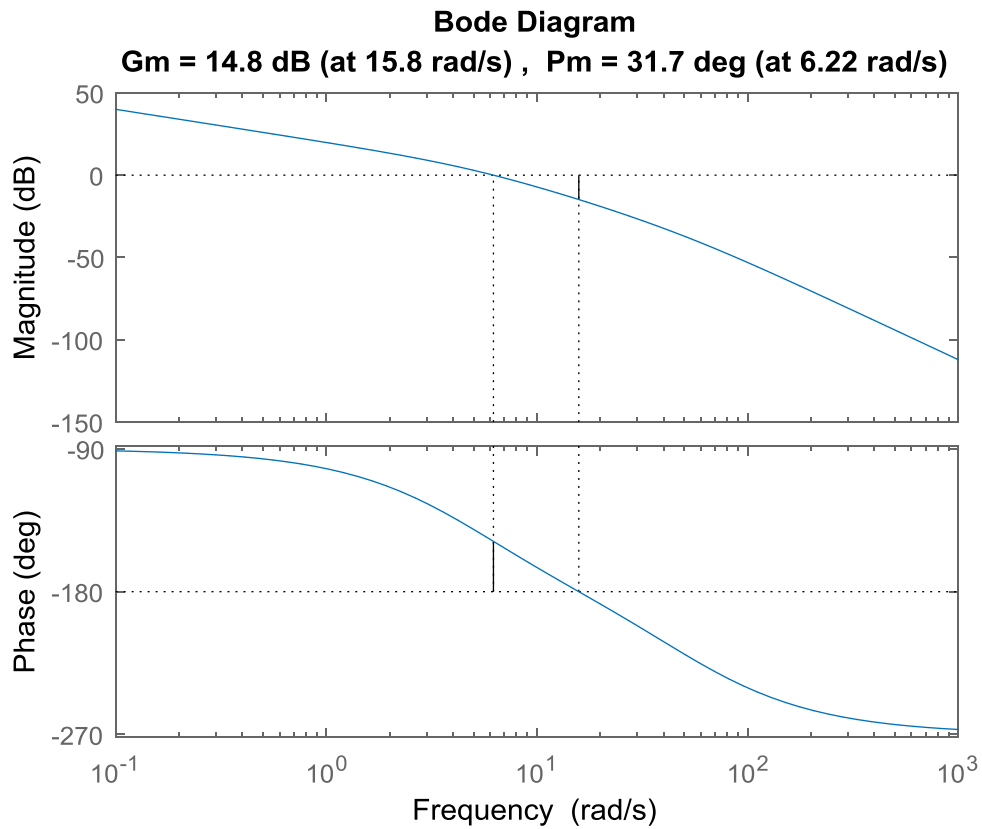
Approx results:

PM of 27.2° at gain crossover frequency of 7.07 rds/s
GM of 14 dB at phase crossover frequency of 15.81 rds/s

Matlab results:

PM of 31.7° at gain crossover frequency of 6.22 rds/s
GM of 14.8 dB at phase crossover frequency of 15.8 rds/s

Problem 3:

Taken from notes:

Example 2:

Sketch the asymptotic Bode plot for the following loop gain.

Annotate the plots completely:
1) Show the values of all break frequencies for magnitude and phase,
2) For magnitude plots: show i) gain along all straight line segments, and ii) slopes,
3) For phase plots: show the slopes.

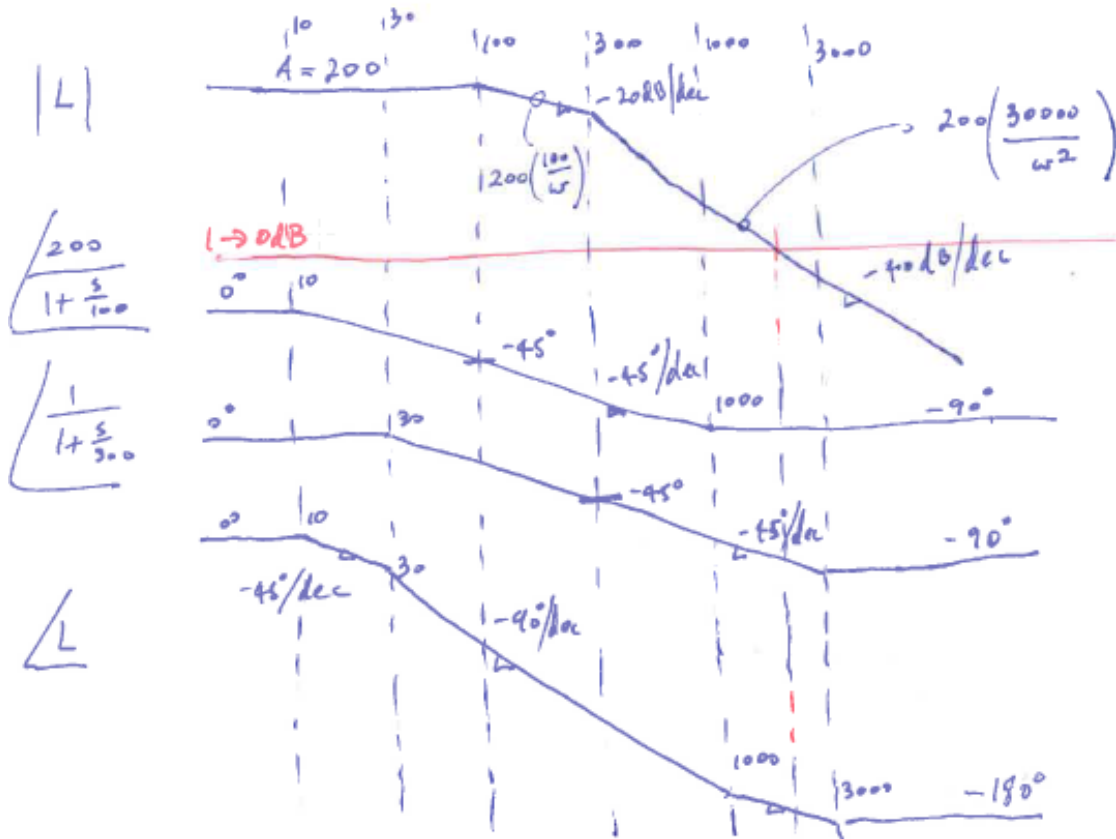$$L(s) = \frac{A}{\left(1 + \frac{s}{\omega_0}\right)\left(1 + \frac{s}{\omega_1}\right)}$$

where:

$$A = 200, \quad \omega_0 = 100, \quad \omega_1 = 300$$

We will also determine the phase and gain margins and associated frequencies.

Problem 3 solution:

$$L(s) = \frac{200}{\left(1 + \frac{s}{100}\right)\left(1 + \frac{s}{300}\right)}$$

$|L|$

$A = 200$ ... $-20dB/dec$

$200\left(\frac{30000}{\omega^2}\right)$

$200\left(\frac{100}{\omega}\right)$

$1 \to 0dB$

$-40dB/dec$

$\frac{200}{1 + \frac{s}{100}}$ ... $-45°$ ... $-45°/dec$ ... 1000 ... $-90°$

$\frac{1}{1 + \frac{s}{300}}$ ... $-45°$ ... $-45°/dec$ ... $-90°$

$\angle L$ ... $-45°/dec$ ... $-90°/dec$ ... 1000 ... 3000 ... $-180°$

NOTE: $\angle L$ exact $= -\operatorname{atan}\left(\frac{\omega}{100}\right) - \operatorname{atan}\left(\frac{\omega}{300}\right)$

Unity gain crossover frequency ω_g

$\Rightarrow 200\left(\frac{30,000}{\omega_g^2}\right) = 1 \Rightarrow \omega_g = \sqrt{200 \times 30,000} = 1000\sqrt{6} = 2450$

$\angle L \Big|_{\omega_g = 2450} = -\operatorname{atan}\left(\frac{2450}{100}\right) - \operatorname{atan}\left(\frac{2450}{300}\right)$

$< -87.67° - 83.01°$

$= -170.68°$

$\Rightarrow$ PHASE MARGIN $= 180 - 168 = 9.32°$ $\left(\Rightarrow \text{STABLE IN CLOSED LOOP}\right)$

PHASE DOES NOT CROSS $-180°$ $\Rightarrow$ ∞ $-180°$ PHASE CROSSOVER FREQUENCY

AND ∞ GAIN MARGIN

Compare the above results with the exact results from Matlab:

```
s=tf('s');
A = 200
w0 = 100
w1 = 300
sys = A/((1+s/w0)*(1+s/w1))
margin(sys)
```

Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 9.36 deg (at 2.44e+03 rad/s)
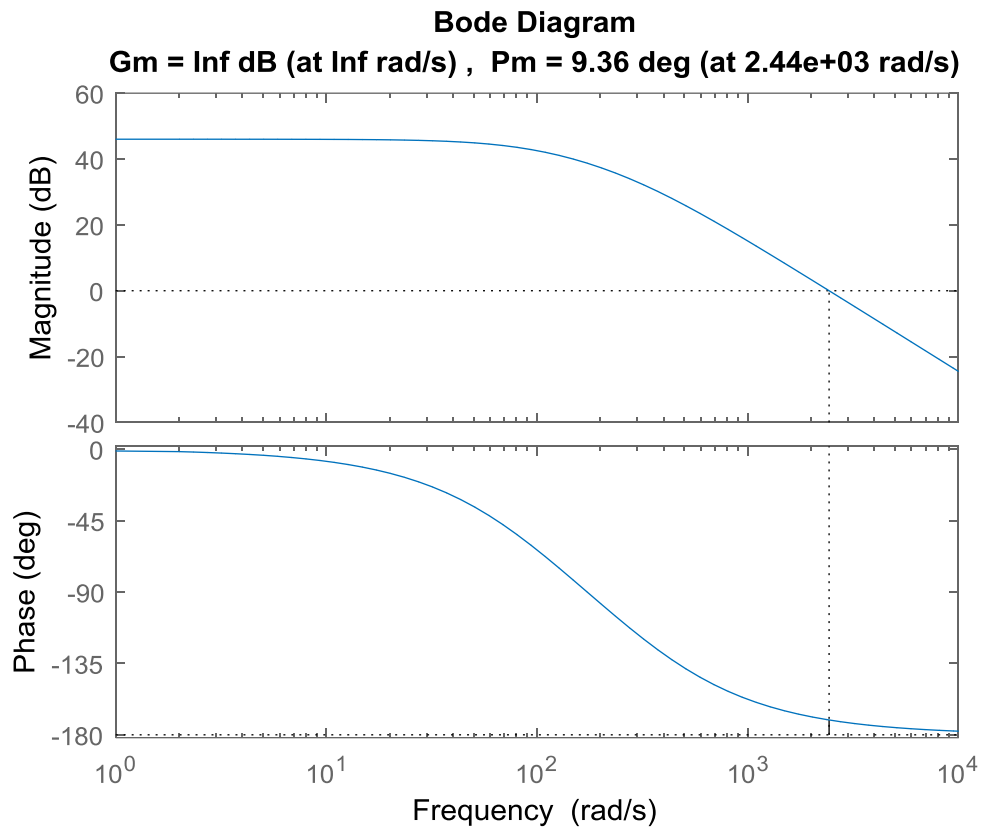
Approx results:

PM of 9.32° at gain crossover frequency of 2450 Hz
GM of ∞ dB at phase crossover frequency of ∞ Hz

Matlab results:

PM of 9.36° at gain crossover frequency of 2440 Hz
GM of ∞ dB at phase crossover frequency of ∞ Hz

Problem 4:

Taken from notes:

Example 3:

Sketch the asymptotic Bode plot for the following loop gain.
Annotate the plots completely and sketch using frequency in Hz (not rds/s):
1) Show the values of all break frequencies for magnitude and phase,
2) For magnitude plots: show i) gain along all straight line segments, and ii) slopes,
3) For phase plots: show the slopes.

$$L(s) = \frac{A\left(1 - \dfrac{s}{\omega_z}\right)}{1 + \dfrac{s}{Q\omega_0} + \left(\dfrac{s}{\omega_0}\right)^2}$$

where:

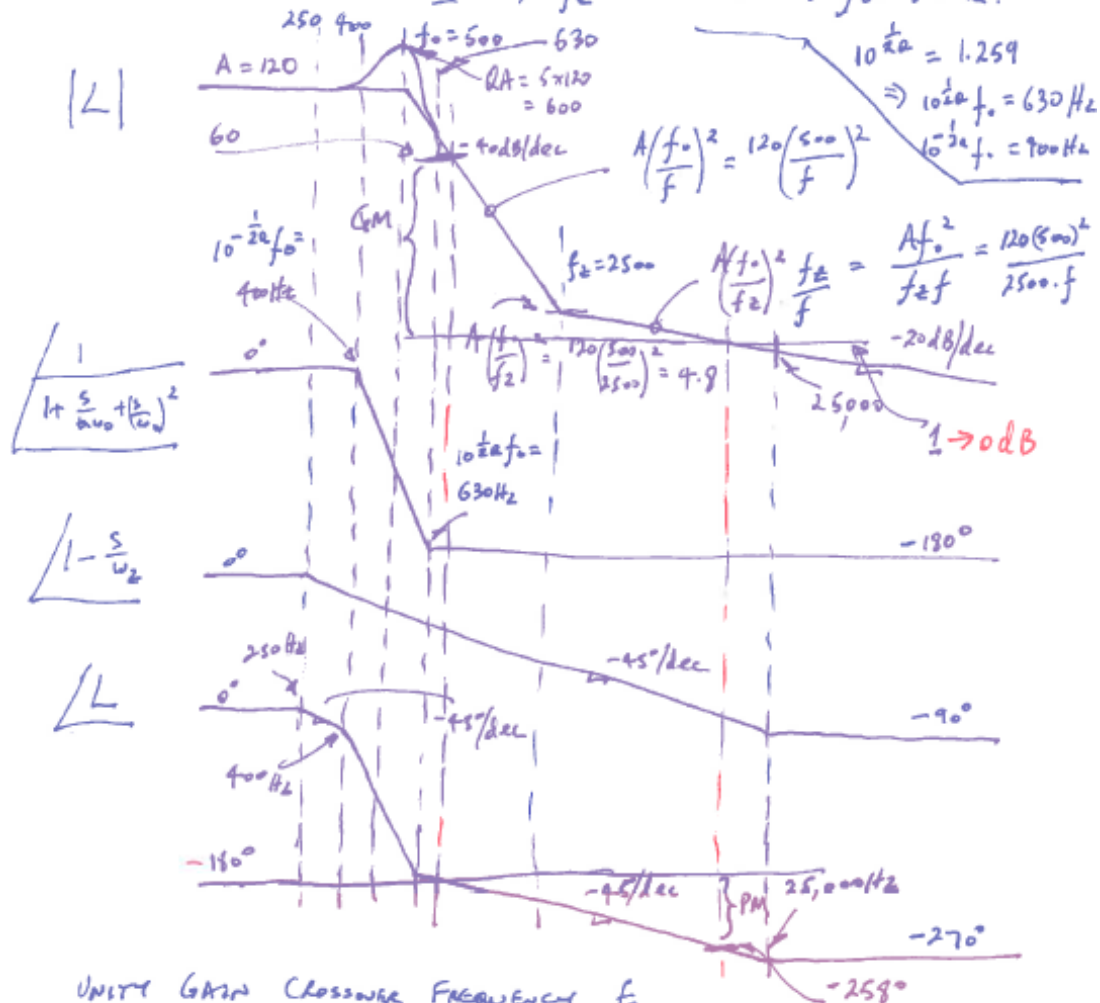$$A = 120, \quad \omega_z = 2\pi(2500), \quad \omega_0 = 2\pi(500), \quad Q = 5$$

We will also determine the phase and gain margins and associated frequencies. All frequencies are given in Hz.

Problem 4 solution:

$$L(s) = \frac{A\left(1 - \frac{s}{\omega_z}\right)}{1 + \frac{s}{Q\omega_o} + \left(\frac{s}{\omega_o}\right)^2} \quad , \quad \begin{array}{l} A = 120 \\ \omega_z = 2\pi(2500) \\ \omega_o = 2\pi(500), Q = 5 \end{array}$$

SKETCH IN Hz $\Rightarrow f_z = 2500$ Hz AND $f_o = 500$ Hz.

$$10^{\frac{1}{2Q}} = 1.259$$
$$\Rightarrow 10^{\frac{1}{2Q}} f_o = 630\,Hz$$
$$10^{-\frac{1}{2Q}} f_o = 900\,Hz$$

$A = 120$ $f_o = 500$ 630

$QA = 5 \times 120 = 600$

$N = 40\,dB/dec$

$A\left(\frac{f_o}{f}\right)^2 = 120\left(\frac{500}{f}\right)^2$

$10^{-\frac{1}{2Q}} f_o^2 = 40\,Hz$ GM

$A\left(\frac{f_o}{f_z}\right)^2 \frac{f_z}{f} = \frac{A f_o^2}{f_z f} = \frac{120(500)^2}{2500 \cdot f}$

$f_z = 2500$

$A\left(\frac{f_o}{f_z}\right)^2 = 120\left(\frac{500}{2500}\right)^2 = 4.9$

$-20\,dB/dec$

25000

$1 \to 0\,dB$

$10^{\frac{1}{2Q}} f_o = 630\,Hz$

$-180°$

$-45°/dec$

$-90°$

250 Hz

$-45°/dec$

400 Hz

$-180°$

$-45/dec$ PM   25,000 Hz

$-270°$

$-258°$

UNITY GAIN CROSSOVER FREQUENCY, f_g

$$\Rightarrow \quad \frac{A f_o^2}{f_z f_g} = 1 \quad \Rightarrow \quad \frac{120(500)^2}{2500} = f_g = 12,000$$

$$\text{PHASE } @ f_g = -\text{atan}\left[\frac{\frac{f_g}{Qf_o}}{1 - \left(\frac{f_g}{f_o}\right)^2}\right] - \text{atan}\left(\frac{f_g}{f_z}\right)$$

$$= -180 - \underbrace{\text{atan}\left(\frac{12,000}{2,500}\right)}_{78°} = -258°$$

$$PM = -78°$$

$PM < 0 \Rightarrow$ UNSTABLE IN CLOSED LOOP.

GAIN MARGIN:

$$L(s) \text{ PHASE} = -\text{atan}\left[\frac{\frac{f}{Qf_0}}{1-\left(\frac{f}{f_0}\right)^2}\right] - \text{atan}\left(\frac{f}{f_2}\right)$$

WE WILL FIND AT WHICH $f = f_p$, PHASE $= -180$

WE WILL USE THE FOLLOWING IDENTITY TO DO THIS:

$$\text{ATAN}(A) + \text{ATAN}(B) = \text{ATAN}\left(\frac{A+B}{1-A\cdot B}\right)$$

$$\Rightarrow \quad \text{ATAN}\left(\frac{\left[\dfrac{\frac{f_p}{5.500}}{1-\left(\frac{f_p}{500}\right)^2}\right] + \dfrac{f_p}{2500}}{1-\left(\dfrac{\frac{f_p}{5.500}}{1-\left(\frac{f_p}{500}\right)^2}\right)\left(\frac{f_p}{2500}\right)}\right) = 180°$$

TAKING TAN OF BOTH SIDES $\Rightarrow$ $\left(\text{since } \tan(180°) = 0\right)$

$$\frac{\frac{f_p}{2500}}{1-\left(\frac{f_p}{500}\right)^2} + \frac{f_p}{2500} = 0.$$

$$\Rightarrow \quad \frac{\frac{f_p}{2500}}{1-\left(\frac{f_p}{500}\right)^2} = \frac{-f_p}{2500} \quad \Rightarrow \quad \frac{1}{1-\left(\frac{f_p}{500}\right)^2} = -1$$

$$\Rightarrow -1 = 1-\left(\frac{f_p}{500}\right)^2 \quad \Rightarrow \quad 2 = \left(\frac{f_p}{500}\right)^2 \quad \Rightarrow \quad f_p = \sqrt{2 \times (500)^2} = 500\sqrt{2}$$
$$= 707 \text{ Hz}$$

GAIN @ $f = f_p = 707$ IS $A\left(\frac{f_0}{f_p}\right)^2 = 120\left(\frac{500}{707}\right)^2 = 60$

GAIN MARGIN $= 20 \log_{10} \frac{1}{|L(jf_p)|} = 20 \log_{10} \frac{1}{60} = -35.6 \text{ dB}$

Compare the above results with the exact results from Matlab:

```
s=tf('s');
A = 120
wz = 2*pi*2500;
w0 = 2*pi*500;
Q = 5;
sys3 = A*(1-s/wz)/((1+s/(Q*w0)+(s/w0)^2))
margin(sys3)
h = gcr;
h.AxesGrid.Xunits = 'Hz'; % display the frequency in Hz
```


Bode Diagram
Gm = -41.6 dB (at 708 Hz) , Pm = -78 deg (at 1.23e+04 Hz)
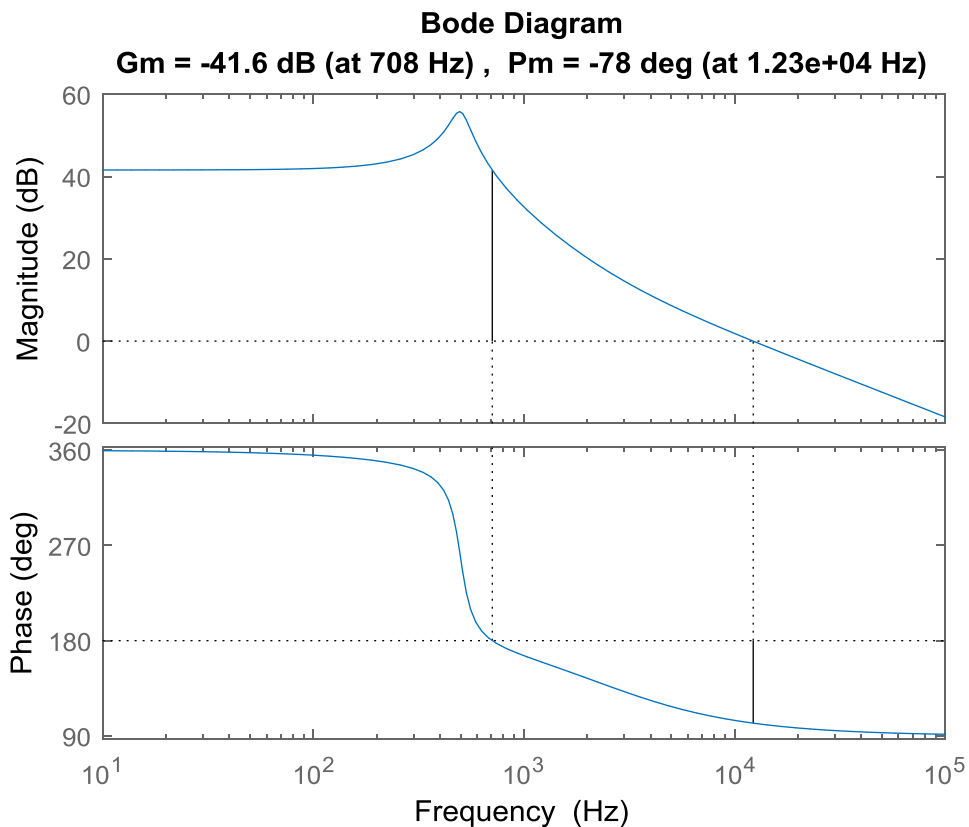
Approx results:

PM of -78° at gain crossover frequency of 12 kHz
GM of -36.6 dB at phase crossover frequency of 707 Hz

Matlab results:

PM of -78° at gain crossover frequency of 12 kHz
GM of -41.6 dB at phase crossover frequency of 708 Hz

Problem 5:

Given the following loop gain:

$$T(s) = \frac{A\left(1+\dfrac{s}{\omega_1}\right)}{1+\dfrac{s}{Q\omega_2}+\left(\dfrac{s}{\omega_2}\right)^2}$$

where

$$
\begin{aligned}
A &= 50 \\
\omega_1 &= 150 \text{ rds/s} \\
\omega_2 &= 8 \text{ rds/s} \\
Q &= 3
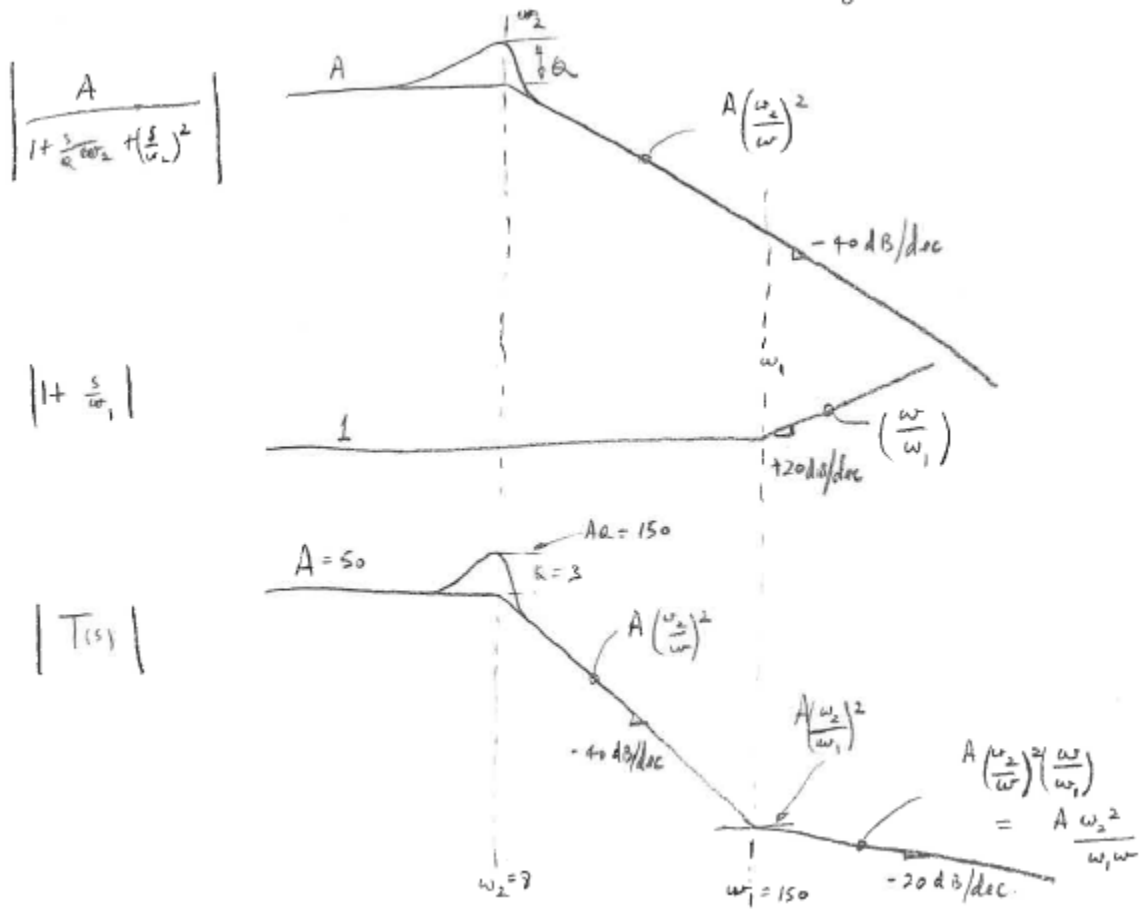\end{aligned}
$$

Using asymptotic approximations only,

a) Sketch the Bode magnitude and phase plots. Be sure to label all break frequencies, slopes of sloping line, gains of sloping lines and gain and phase levels on zero slope lines.

b) Find the maximum gain (as an absolute value) and the frequency or range of frequencies at which it occurs.

c) Using your plots determine the phase margin and associated crossover frequency.

d) Using your plots determine the gain margin and associated crossover frequency.

e) Determine whether the closed loop system is stable.

Problem 5 Solution:

$$T(s) = \frac{A\left(1 + \dfrac{s}{\omega_1}\right)}{1 + \dfrac{s}{Q\omega_2} + \left(\dfrac{s}{\omega_2}\right)^2}$$

$A = 50$
$\omega_1 = 150$ rds/s
$\omega_2 = 8$ rds/s
$Q = 3$

a)

$$\angle \frac{A}{1 + \frac{s}{Qw_2} + \left(\frac{s}{w_2}\right)^2}$$

$10^{-\frac{1}{2R}} w_2 = \frac{8}{1.47} = 5.5$

$10^{\frac{1}{2R}} = 1.47$

$w_2 = 8$

$0°$

$10^{\frac{1}{2R}} w_2 = 1.47 \times 8 = 11.7$

$-180°$

$$\angle 1 + \frac{s}{w_1}$$

$10 w_1$

$+90°$

w_1

$\frac{w_1}{10} = 15$

$+45°/dec$

$0°$

$5.5 \quad w_2 = 8 \quad 11.7 \quad \frac{w_1}{10} = 15 \quad w_1 = 150 \quad 10 w_1 = 1500$

$$\angle T(s)$$

$0°$

-90

$-180°$

$+45°/dec$

-90

$5.5 \quad w_2 = 8 \quad 11.7 \quad \frac{w_{1/10}}{15} = 15 \quad w_1 = 150 \quad 10 w_1 = 1500$

$$T(s) = \frac{A\left(1 + \frac{s}{\omega_1}\right)}{1 + \frac{s}{Q\omega_2} + \left(\frac{s}{\omega_2}\right)^2}$$
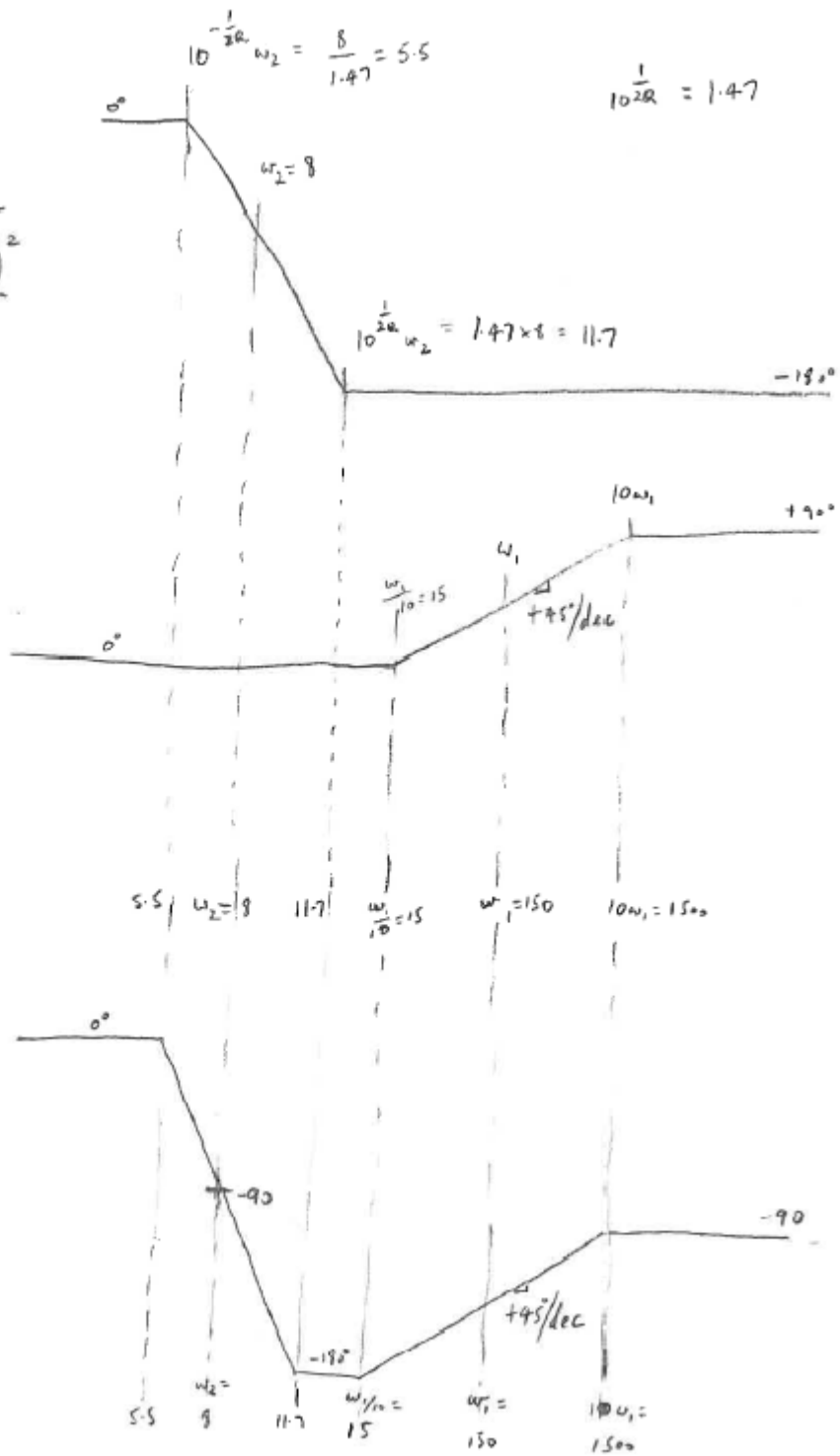
$A = 50$

$\omega_1 = 150$

$\omega_2 = 8$

$Q = 3$

$|T(s)|$

$\omega_2 = 8$

$AQ = 150$

$A = 50$

$Q = 3$

$A\left(\frac{\omega_2}{\omega}\right)^2$

$\omega_1 = 150$

$-40 dB/dec$

$A\left(\frac{\omega_2}{\omega_1}\right)^2$

$\frac{A\omega_2^2}{\omega_1 \omega}$

$-20 dB/dec$

$\angle T(s)$

$0°$

$10^{-\frac{1}{2Q}}\omega_2 = 5.5$

$10^{\frac{1}{2Q}}\omega_2 = 11.7$

$-90°$

$-90°$

$-180°$

$45°/dec$

5.5 $\omega_2 = 8$ 11.7 $\frac{\omega_1}{10} = 15$ $\omega_1 = 150$ $10\omega_1 = 1500$
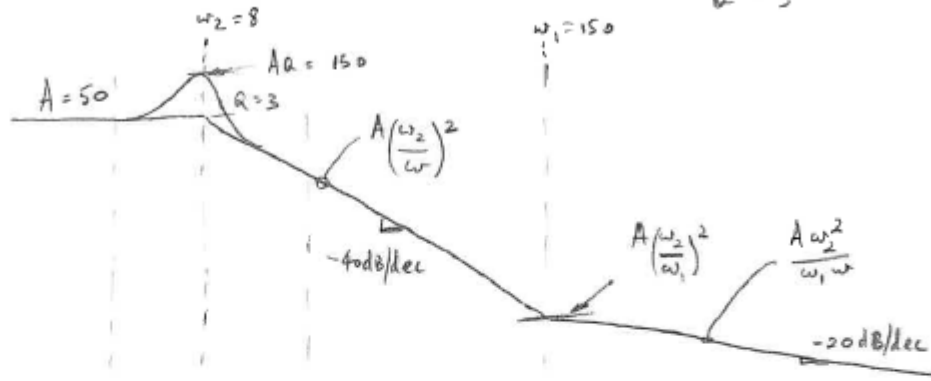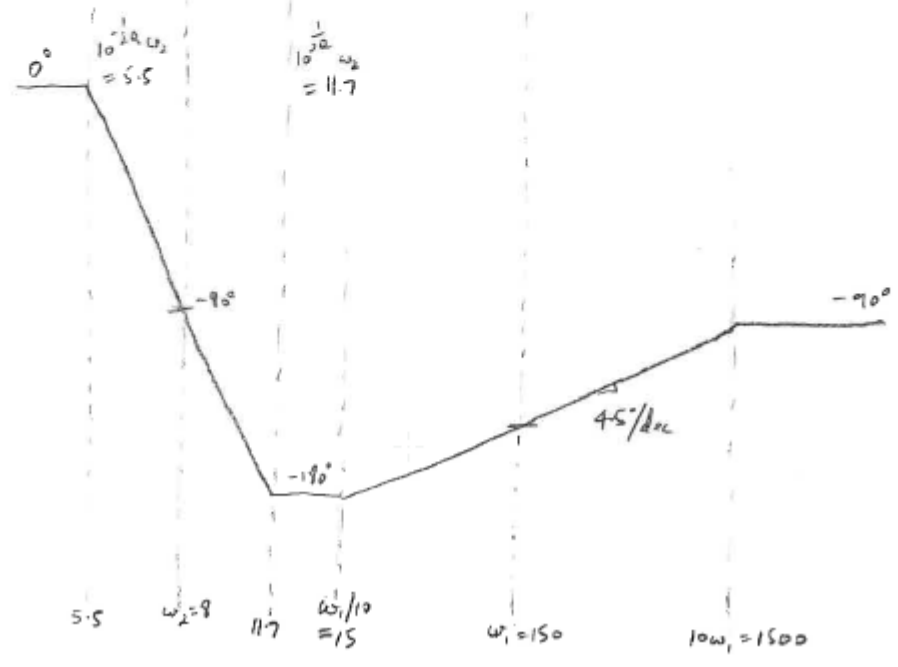
b) MAX GAIN OCCURS AT $\omega_2 = 8$ AND IS

GIVEN BY $QA = 3 \times 50 = 150$

c) AT $\omega_1 = 150$ GAIN IS $A\left(\frac{\omega_2}{\omega_1}\right)^2 = 50\left(\frac{8}{150}\right) = 0.14 < 1$

$\Rightarrow$ UNITY GAIN OCCURS AT $\omega_c < \omega_1$

$\Rightarrow$ $A\left(\frac{\omega_2}{\omega_c}\right)^2 = 1$ $\Rightarrow$ $\omega_c = \omega_2\sqrt{A} = 8\sqrt{10} = 56.6 \text{ rds/s}$

EXACT PHASE ϕ IS GIVEN BY

$$\phi = -\tan^{-1}\left[\frac{\frac{1}{Q}\frac{\omega}{\omega_2}}{1 - \left(\frac{\omega}{\omega_2}\right)^2}\right] + \tan^{-1}\left(\frac{\omega}{\omega_1}\right), \quad \forall \omega$$

$\underbrace{\qquad\qquad\qquad\qquad}$
$\approx -180°$ for $\omega > 11.7$ rds/s (see phase plot)

$\Rightarrow$ $\phi_c \approx -180° + \tan^{-1}\left(\frac{\omega_c}{\omega_1}\right)$

$\approx -180° + \tan^{-1}\left(\frac{56.6}{150}\right)$

$\approx -160°$

$\Rightarrow \phi_m =$ PHASE MARGIN $= 180° - 160°$

$= 20°$

d) ∞ GAIN MARGIN, NO ASSOCIATED FREQUENCY

e) $\phi_m > 0$ $\Rightarrow$ STABLE

Exact results from Matlab:

```
s=tf('s')
A = 50
w1 = 150
w2 = 8
Q = 3
sys = A*(1+s/w1)/(1+s/(Q*w2)+(s/w2)^2)
margin(sys)
```

Bode Diagram
Gm = Inf , Pm = 24.2 deg (at 59.2 rad/s)

System: sys
Frequency (rad/s): 7.77
Magnitude (dB): 43.7
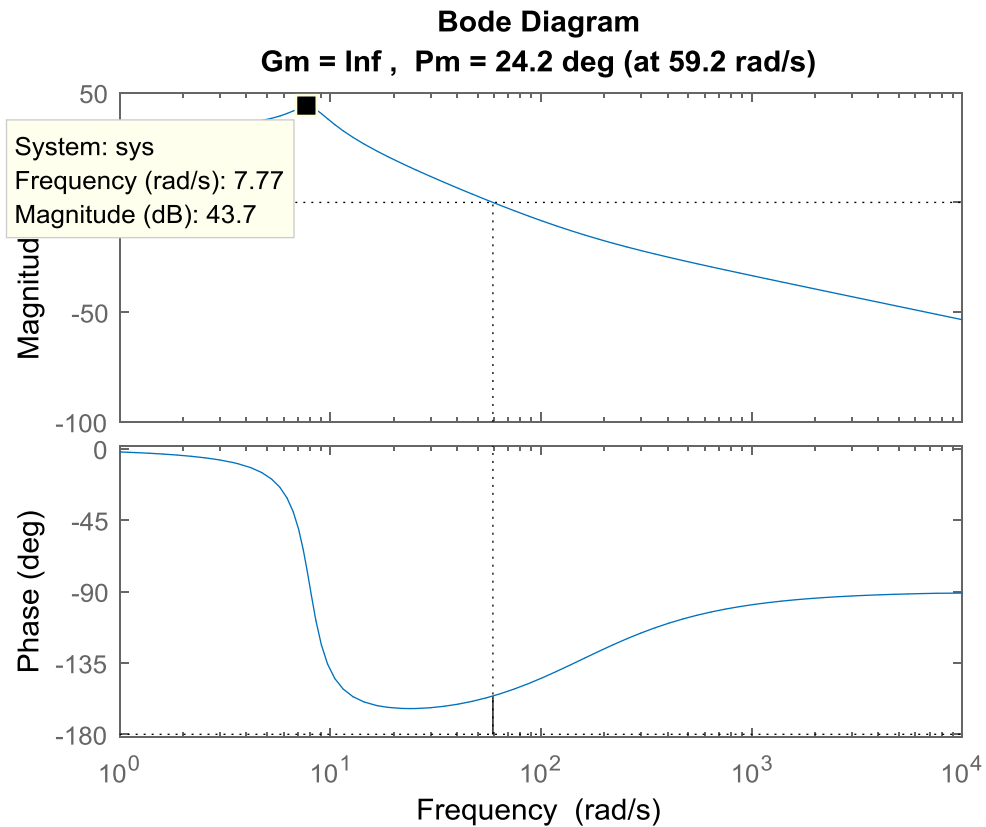
Note that the maximum gain of 43.7 dB corresponds to a gain of 153 ($= 10^{\frac{43.7}{20}}$).

Approx results:

        PM of 20° at gain crossover frequency of 56.6 rds/s

        GM of ∞ dB at phase crossover frequency of ∞ Hz

        Max gain of 150 at 8 rds/s

Matlab results:

        PM of 24.2° at gain crossover frequency of 59.2 rds/s

        GM of ∞ dB at phase crossover frequency of ∞ Hz

        Max gain of 153 at 7.77 rds/s

Problem 6:

Given the following loop gain:

$$T(s) = \frac{A\left(1 - \dfrac{s}{\omega_z}\right)}{1 + \dfrac{s}{Q\omega_0} + \left(\dfrac{s}{\omega_0}\right)^2}$$

where

$$A = 9$$
$$\omega_z = 7000 \text{ rds/s}$$
$$\omega_0 = 1000 \text{ rds/s}$$
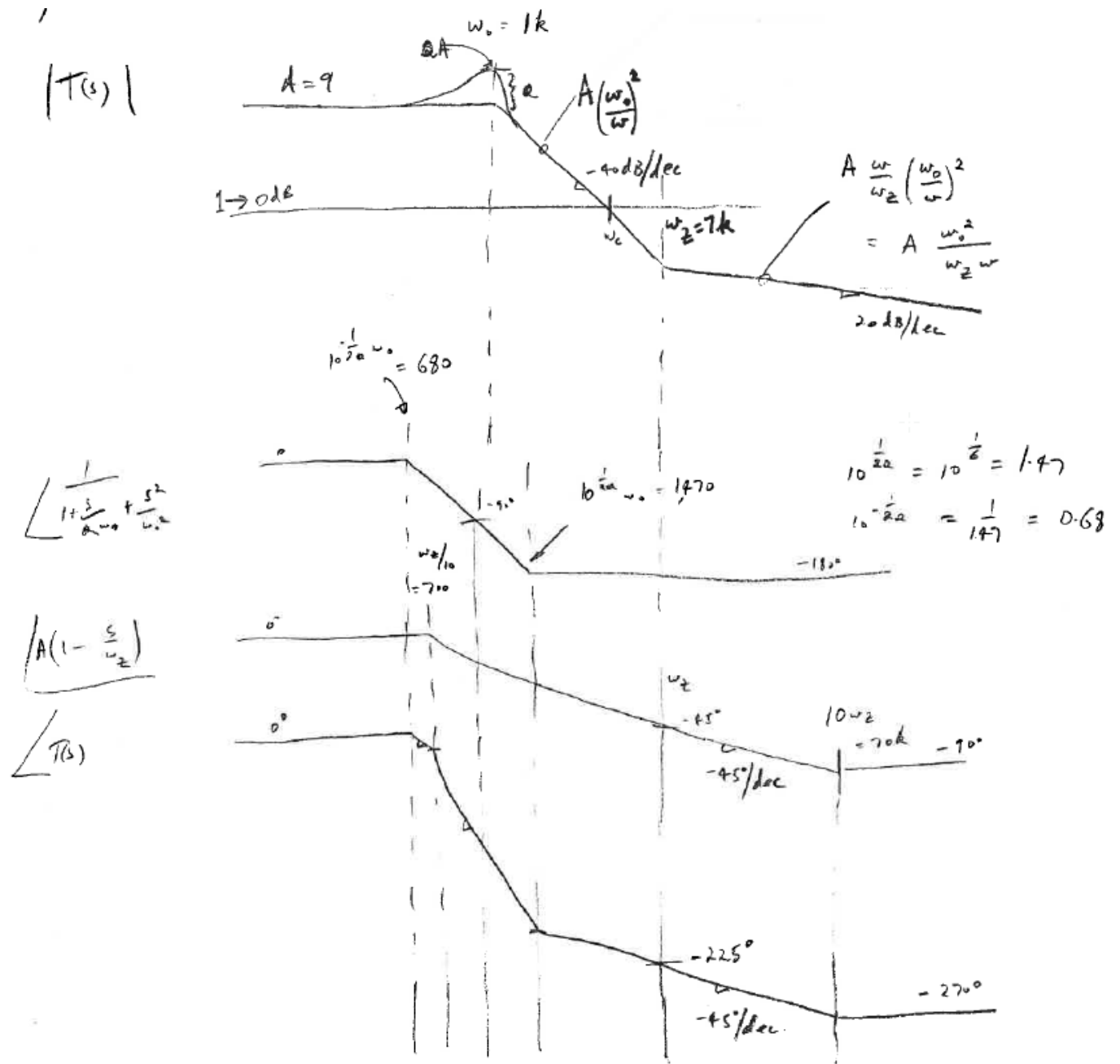$$Q = 3$$

Using asymptotic approximations only,

a) Sketch the Bode magnitude and phase plots. Be sure to label all break frequencies, slopes of sloping line, gains of sloping lines and gain and phase levels on zero slope lines.

b) Find the maximum gain (as an absolute value) and the frequency or range of frequencies at which it occurs.

c) Using your plots determine the phase margin (expressed in degrees) and associated crossover frequency, ω_c.

d) Using your plots determine the gain margin (expressed in dB) and associated crossover frequency, ω_M.

(If needed,   Hint: $\tan(X + Y) = \dfrac{\tan X + \tan Y}{1 - \tan X \tan Y}$)

e) Determine whether the closed loop system is stable.

Problem 6 Solution:

a)

$|T(s)|$

$w_0 = 1k$

ωA

$A = 9$

$A\left(\frac{w_0}{w}\right)^2$

$\frac{2}{3}a$

$-40dB/dec$

$1 \to 0dB$

$w_z = 7k$

ω_c

$A \frac{w}{w_z}\left(\frac{w_0}{w}\right)^2$

$= A \frac{w_0^2}{w_z w}$

$20dB/dec$

$10^{\frac{1}{2a}} w_0 = 680$

$\angle \frac{1}{1 + \frac{s}{aw_0} + \frac{s^2}{w_0^2}}$

$10^{\frac{1}{2a}} = 10^{\frac{1}{6}} = 1.47$

$10^{-\frac{1}{2a}} = \frac{1}{1.47} = 0.68$

$6^{\frac{1}{2a}} w_0 = 1470$

$w_z/10$
$= 700$

$-180°$

$\angle A\left(1 - \frac{s}{w_z}\right)$

$0°$

w_z

$-45°$

$10 w_z$
$= 70k$

$-90°$

$\angle T(s)$

$0°$

$-45°/dec$

$-225°$

$-270°$

$-45°/dec$

b) MAX GAIN = $QA = 3 \times 9 = 27$ @ $w_o = 1k \, rds/sec$.

c) $A\left(\dfrac{w_o}{w_c}\right)^2 = 1$ $\Rightarrow$ $w_c = w_o \sqrt{A}$

                                 $= 1000 \times \sqrt{9}$

w_c is the crossover $= 3,000$

     phase $= -180 - \tan^{-1}\left(\dfrac{w_c}{w_2}\right)$ for $w_c \geqslant 10^{\frac{1}{2Q}} w_o$

         $= -180 - \tan^{-1}\left(\dfrac{3000}{7000}\right)$

         $= -180 - 23.2$

         $= -203.2$

         $\Rightarrow$ phase margin $= \underline{-23.2^{\circ}}$

d)   exact phase $\forall w = -\tan^{-1}\left[\dfrac{\frac{1}{Q}\frac{w}{w_o}}{1 - \left(\frac{w}{w_o}\right)^2}\right] - \tan^{-1}\left(\dfrac{w}{w_c}\right)$

   at $w = w_m$ (phase crossover frequency) , phase $= -180$

   $\Rightarrow$ $-180 = -\tan^{-1}\left[\dfrac{\frac{1}{Q}\frac{w_m}{w_o}}{1 - \left(\frac{w_m}{w_o}\right)^2}\right] - \tan^{-1}\left(\dfrac{w_m}{w_c}\right)$

   $\Rightarrow$ $180 = \tan^{-1}\left[\dfrac{\frac{1}{Q}\frac{w_m}{w_o}}{1 - \left(\frac{w_m}{w_o}\right)^2}\right] + \tan^{-1}\left(\dfrac{w_m}{w_c}\right)$ ($*$)

use calculator to solve for w_m $\underline{\underline{OR}}$

   use $\tan(X+Y) = \dfrac{\tan X + \tan Y}{1 - \tan X \tan Y}$

       let $X = \tan^{-1}\left[\dfrac{\frac{1}{Q}\frac{w_m}{w_o}}{1 - \left(\frac{w_m}{w_o}\right)^2}\right]$, $Y = \tan^{-1}\left(\dfrac{w_m}{w_c}\right)$

$(*) \Rightarrow$ $\underbrace{\tan(180)}_{0} = \tan(x + y)$

$\Rightarrow$ $0 = \tan\left\{\tan^{-1}\left[\dfrac{\frac{1}{a}\frac{\omega_m}{\omega_0}}{1 - \left(\frac{\omega_m}{\omega_0}\right)^2}\right]\right\} + \tan\left\{\tan^{-1}\left(\frac{\omega_m}{\omega_z}\right)\right\}$

$= \dfrac{\frac{1}{a}\frac{\omega_m}{\omega_0}}{1 - \left(\frac{\omega_m}{\omega_0}\right)^2} + \dfrac{\omega_m}{\omega_z}$

$1 + \dfrac{\omega_z}{a\omega_0} = \left(\dfrac{\omega_m}{\omega_0}\right)^2$

$\Rightarrow \left(-\dfrac{1}{\omega_z}\right)\left(1 - \left(\dfrac{\omega_m}{\omega_0}\right)^2\right) = \dfrac{1}{a\omega_0}$

$\Rightarrow \omega_m = \omega_0\sqrt{1 + \dfrac{\omega_z}{a\omega_0}}$

$\Rightarrow \quad 1 - \left(\dfrac{\omega_m}{\omega_z}\right)^2 = -\dfrac{\omega_z}{a\omega_0}$

$a = 3, \ \omega_0 = 1{,}000, \ \omega_z = 7000$

$\Rightarrow$

$\omega_m = 1{,}830 \ \text{rds/s.}$

magnitude at $\omega_m = A\left(\dfrac{\omega_0}{\omega_m}\right)^2$
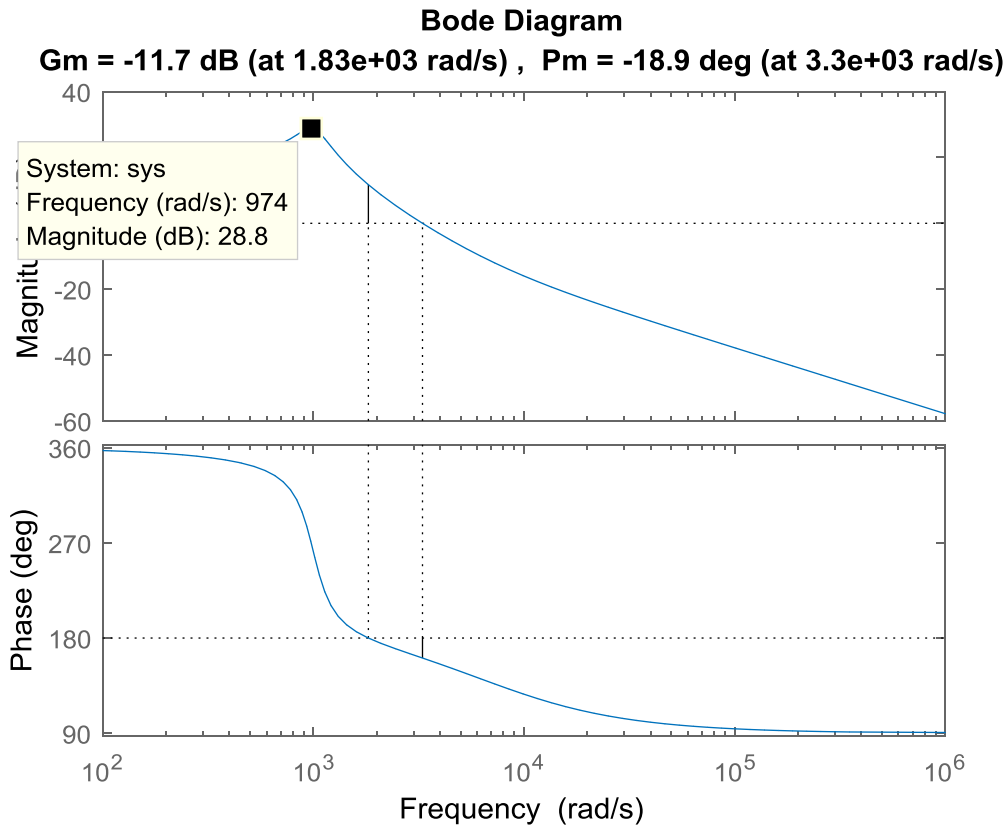
$= 9\left(\dfrac{1000}{1830}\right)^2$

$= 2.69 \quad \Rightarrow \quad 8.6 \ dB$

$\Rightarrow$ gain magn$^\circ$ = $-8.6 \ dB$

e)   UNSTABLE, since $Pm < 0$

Exact results from Matlab:

```
s=tf('s')
A = 9
w1 = 7000
w2 = 1000
Q = 3
sys = A*(1-s/w1)/(1+s/(Q*w2)+(s/w2)^2)
margin(sys)
```

Bode Diagram
Gm = -11.7 dB (at 1.83e+03 rad/s) , Pm = -18.9 deg (at 3.3e+03 rad/s)

Note that the maximum gain of 28.8 dB corresponds to a gain of 27.5 ($= 10^{\frac{28.8}{20}}$).

Approx results:

    PM of $-23.2°$ at gain crossover frequency of 3000 rds/s
    GM of -8.6 dB at gain crossover frequency of 1830 rds/s
    Max gain of 27 at 1000 rds/s

Matlab results:

    PM of $-18.9°$ at gain crossover frequency of 3300 rds/s
    GM of -11.7 dB at gain crossover frequency of 1830 rds/s
    Max gain of 27.5 at 974 rds/s

Problem 7:

Given the following transmittance:

$$T(s) = \frac{As}{\left(\dfrac{s^2}{w_o^2} + \dfrac{s}{Qw_o} + 1\right)\left(1 + \dfrac{s}{w_p}\right)}$$

where

$$A = 40$$
$$w_o = 15$$
$$Q = 3$$
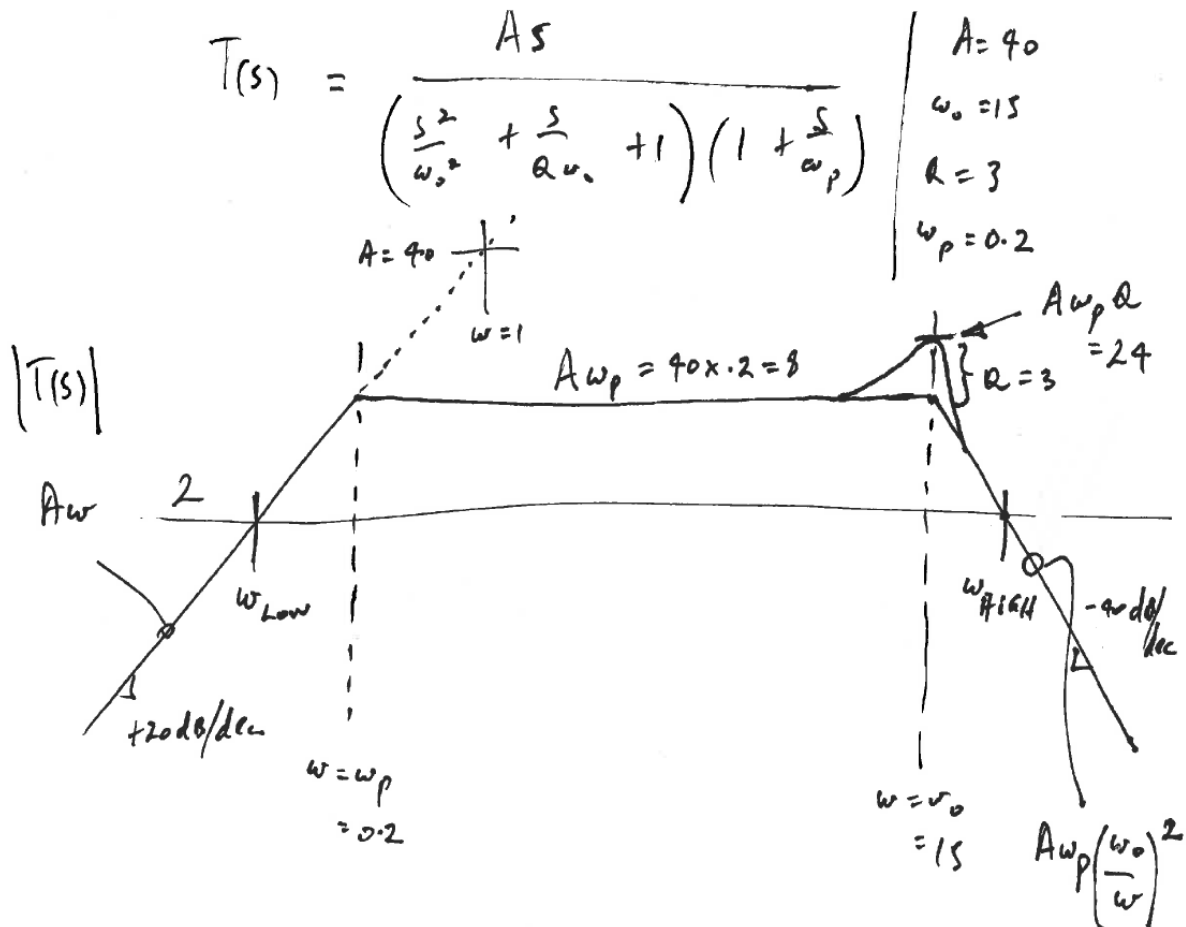$$w_p = 0.2$$

Using an approach that uses asymptotic approximations,

a). Sketch $|T(s)|$, the magnitude characteristic of $T(s)$. Be sure to label the break frequencies, the slopes of sloping lines and gains of constant gain lines.

b). Determine the maximum gain of $T(s)$ (expressed as an absolute value). At what frequency, w, does this maximum occur?

c). Determine the range of frequencies, $w_{low} \le w \le w_{high}$ for which $|T(jw_{low})| \le 2 \le |T(jw_{high})|$. That is, find the range of frequencies where the gain is greater than or equal to 2.

Problem 7 Solution:

a)

$$T_{(s)} = \frac{As}{\left(\frac{s^2}{\omega_o^2} + \frac{s}{Q\omega_o} + 1\right)\left(1 + \frac{s}{\omega_p}\right)} \quad \Bigg| \quad \begin{array}{l} A = 40 \\ \omega_o = 15 \\ Q = 3 \\ \omega_p = 0.2 \end{array}$$

b) MAX GAIN $\approx A\omega_p Q \approx 24$ @ 15 rds/sec
(ω_o)

$$\begin{pmatrix} \text{EXACT RESULTS FROM} \\ \text{MATLAB} \quad \omega = 14.56 \text{ rds/sec.} \\ \text{MAX GAIN} = 24.3376 \end{pmatrix}$$

c) $A\omega_{Low} = 2 \Rightarrow \omega_{Low} = \frac{2}{A} = 0.05 \quad \begin{pmatrix} \text{EXACT RESULT FROM} \\ \text{MATLAB} \quad 0.0516 \end{pmatrix}$

$A\omega_p\left(\frac{\omega_o}{\omega_{HIGH}}\right)^2 = 2 \Rightarrow \omega_{HIGH} = \sqrt{\frac{A\omega_o^2\omega_p}{2}} = 30 \quad \begin{pmatrix} \text{EXACT RESULT FROM} \\ \text{MATLAB} \quad 33.3329 \end{pmatrix}$