Solution to HW #01

#1. $Z_1 = 3+5i$

$$\left.\begin{aligned} |Z_1| &= \sqrt{3^2+5^2} = 5.83 \\ \angle Z_1 &= \tan^{-1}\left(\frac{5}{3}\right) = 59.03^\circ \end{aligned}\right\} \checkmark$$

$$Z_2 = \frac{-5-i}{1+i} = \frac{(-5-i)(1-i)}{(1+i)(1-i)} = \frac{-5-1+4i}{2}$$

$$Z_2 = -3+2i$$

$$\left.\begin{aligned} |Z_2| &= \sqrt{3^2+2^2} = \sqrt{13} \\ \angle Z_2 &= \tan^{-1}\left(\frac{2}{-3}\right) = 146.31^\circ \end{aligned}\right\} \checkmark$$

Another way,

$$Z_2 = \frac{\sqrt{5^2+1^2}\ \tan^{-1}\left(\frac{-1}{-5}\right)}{\sqrt{1^2+1^2}\ \tan^{-1}\left(\frac{1}{1}\right)} = \frac{\sqrt{26}\ \angle 191.31^\circ}{\sqrt{2}\ \angle 45^\circ}$$

$$= \sqrt{13}\ \angle 146.31^\circ \checkmark$$

$Z_3 = 4-5i$

$$\left.\begin{aligned} |Z_3| &= \sqrt{4^2+5^2} = \sqrt{41} \\ \angle Z_3 &= \tan^{-1}\left(\frac{-5}{4}\right) = -51.34^\circ \end{aligned}\right\} \checkmark$$

#2 $f_1 = t^2 + 2t + 5$

$$\mathcal{L}[f_1] = F_1 = \mathcal{L}[t^2] + \mathcal{L}[2t] + \mathcal{L}[5]$$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{5}{s} = \frac{5s^2+2s+2}{s^3} \checkmark$$

$$f_2 = -e^{-2t}\cos 3t$$

$$\mathcal{L}[\cos 3t] = \frac{s}{s^2+9}$$

$$\rightarrow \mathcal{L}\left[-e^{-2t}\cos 3t\right] = -\frac{(s+2)}{(s+2)^2+3^2} \checkmark$$

#3. $F_1 = \frac{s}{(s+2)^2+16} = \frac{s+2}{(s+2)^2+16} - \frac{2}{(s+2)^2+16}$

$$f_1 = \mathcal{L}^{-1}[F_1] = e^{-2t}\left[\cos 4t - \frac{1}{2}\sin 4t\right] \checkmark$$

$$F_2 = \frac{2s^2+3s-5}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+2)}$$

$$\rightarrow A(s+1)(s+2) + Bs(s+2) + Cs(s+1) = 2s^2+3s-5$$

$s=0 \rightarrow 2A = -5$ or $A = -5/2$

$s=-1 \rightarrow -B = -6$ or $B = 6$

$s=-2 \rightarrow 2C = -3$ or $C = -3/2$

$$F_2 = \frac{-5/2}{s} + \frac{6}{s+1} + \frac{-3/2}{s+2}$$

$$\longrightarrow f_2 = -\frac{5}{2} + 6e^{-t} - \frac{3}{2}e^{-2t} \quad \checkmark$$

$$F_3 = \frac{3s}{(s+2)^2(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2} + \frac{C}{(s+3)}$$

or $A(s+2)(s+3) + B(s+3) + C(s+2)^2 = 3s$

or $s^2(A+C) + s(5A+B+4C) + (6A+3B+4C) = 3s$

or
$$\left.\begin{array}{l} A+C=0 \\ 5A+B+4C=3 \\ 6A+3B+4C=0 \end{array}\right\} \quad \left.\begin{array}{l} A+B=3 \\ 2A+3B=0 \end{array}\right\} \quad \begin{array}{l} A=9 \\ B=-6 \\ C=-9 \end{array}$$

$$F_3 = \frac{9}{s+2} - \frac{6}{(s+2)^2} - \frac{9}{(s+3)}$$

$$f_3 = e^{-2t}(9-6t) - 9e^{-3t}$$

#4. $\ddot{y} - 3\dot{y} + 2y = 4 \qquad y(0)=0,\ \dot{y}(0)=-1$

$$\Big\downarrow \mathcal{L}$$

$$s^2Y - s\underbrace{y(0)}_{0} - \dot{y}(0) - 3[sY - \underbrace{y(0)}_{0}] + 2Y = \frac{4}{s}$$

$$(s^2-3s+2)\,Y(s) = \frac{4}{s} + \dot{y}(0) = \frac{4}{s} - 1 = \frac{4-s}{s}$$

$$Y(s) = \frac{(4-s)}{s(s^2-3s+2)} = \frac{(4-s)}{s(s-1)(s-2)}$$

Let $Y(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$\longleftrightarrow A(s-1)(s-2) + Bs(s-2) + Cs(s-1) = 4-s$

$s=0 \rightarrow 2A = 4$ or $A=2$

$s=1 \rightarrow -B=3$ or $B=-3$

$s=2 \rightarrow 2C=2$ or $C=1$

$$Y(s) = \frac{2}{s} - \frac{3}{s-1} + \frac{1}{s-2}$$

or $y(t) = 2 - 3e^{t} + e^{2t}$ ✓

#5. $F(s) = \frac{3}{s(s^2+2s+10)}$

$sF(s) = \frac{3}{(s^2+2s+10)}$

Roots of denominator polynomial of $sF(s)$:

$s^2+2s+10 = 0 \longrightarrow s_{1,2} = \frac{-2 \pm \sqrt{4-40}}{2}$

$= -1 \pm 3i$

Negative real part

FVT is applicable.

Final value of $f(t)$

$$= \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{3}{(s^2+2s+10)}$$

$$= \frac{3}{10} \checkmark$$

Computing $\lim_{t\to\infty} f(t)$ directly

$$F(s) = \frac{3}{s(s^2+2s+10)} = \frac{A}{s} + \frac{Bs+C}{(s+1)^2+3^2}$$

$$A\left[s^2+2s+10\right] + (Bs+C)s = 3$$

$$s^2(A+B) + s(2A+C) + (10A) = 3$$

or $\left.\begin{aligned} A+B&=0 \\ 2A+C&=0 \\ 10A&=3 \end{aligned}\right\}$ $A = 3/10$

We do not care about B & C values, as will be clear below

$$F(s) = \frac{3/10}{s} + \frac{Bs+C}{(s+1)^2+3^2}$$

$$= \frac{3}{10s} + \frac{B(s+1)}{(s+1)^2+3^2} + \frac{(C-B)}{(s+1)^2+3^2}$$

$$f(t) = \frac{3}{10} + e^{-t}\left[B\cos 3t + \frac{(C-B)}{3}\sin 3t\right]$$

As $t \to \infty$, $e^{-t} \to 0$

Therefore, $\lim_{t \to \infty} f(t) = \frac{3}{10}$ ✓