**EE 520: Random Processes** **Fall 2021**

# Lecture 14
# Filtering Random Processes

*Instructor Name: John Lipor*

*Recommended Reading:* Pishro-Nik: 10.2.3 - 10.2.4; Gubner: 10.5 - 10.8

We continue our discussion of wide-sense stationary (WSS) random processes (RPs) with some additional properties as well as two filters that are used to perform detection and estimation in RPs.

## 1 Power Spectral Density for WSS Random Processes

Recall from Lecture 13 that a RP $\{X_t\}$ is WSS if

1. $\mathbb{E}[X_t] = \mathbb{E}[X_s]$ for all $s, t$ (i.e., the mean does not change over time)
2. $\mathbb{E}[X_t X_s]$ depends on $t, s$ only through their difference $t - s$.

For WSS processes, we write the correlation function as $R_X(\tau)$. We often work with the Fourier transform (FT) of $R_X(\tau)$, which is called the *power spectral density* (PSD)

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau.$$

As motivation for the PSD, note that the energy in a waveform or function may be defined as $\int_{-\infty}^{\infty} |x(t)|^2 \, dt$ when this quantity is finite. For signals with infinite power, we consider the average energy

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt.$$

These (hopefully familiar) quantities can also be analyzed for RPs.

### 1.1 Power in a Process

Consider a WSS process $\{X_t\}$. Note that

$$\int_{-\infty}^{\infty} |X_t|^2 \, dt \quad \text{and} \quad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |X_t|^2 \, dt$$

are both random quantities, so when we talk about the power in a process we quantify the *expected* average power.

**Definition 1.** For a WSS process $\{X_t\}$, the **expected average power** is

$$P_X = \mathbb{E}\left[X_t^2\right] = R_X(0) = \int_{-\infty}^{\infty} S_X(f) df.$$

Note that the last term above is simply the Fourier transform of $R_X$ evaluated at $\tau = 0$. Further, we see that the power in the process can be found by integrating the PSD across all frequencies.

**Example 1.** Find the power in the frequency band $W_1 \le |f| \le W_2$ for the WSS process $\{X_t\}$.

We can model the power in this frequency band as passing $\{X_t\}$ through an LTI system with unit gain over the frequency range $W_1 \le |f| \le W_2$, i.e., we define the filter to have transfer function

$$H(f) = \begin{cases} 1, & W_1 \le |f| \le W_2 \\ 0, & \text{otherwise.} \end{cases}$$

Letting $\{Y_t\}$ denote the output RP (which is still WSS), we can then compute the power as

$$\begin{aligned} P_Y &= \int_{-\infty}^{\infty} S_Y(f) df \\ &= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df \\ &= \int_{-W_2}^{-W_1} S_X(f) df + \int_{W_1}^{W_2} S_X(f) df \\ &= 2\int_{W_1}^{W_2} S_X(f) df, \end{aligned}$$

where the last line follows by the symmetry of $S_X$.

## 1.2 White Noise

A process with constant power across all frequencies is called **white noise**. In this case, we have

$$R_X(\tau) = \sigma^2 \delta(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_X(f) = \sigma^2, \ \forall f.$$

Note that this is a useful mathematical idealization, and the power actually does fall off for high frequencies in practice for all real-world signals. This model causes a few problems:

1. $R_X(0) = \sigma^2 \delta(0)$ is undefined
2. $P_X = \int_{-\infty}^{\infty} \sigma^2 dx = \infty$.

In problems, we sometimes write the noise level $\sigma^2 = N_0/2$, which is common notation in analysis of communication systems.

# 2 The Matched Filter

Suppose we're running a radar system, where we transmit a deterministic signal $v(t)$ and measure a noisy return

$$R_t = \begin{cases} X_t, & \text{no aircraft present (case 0)} \\ v(t) + X_t, & \text{aircraft present (case 1),} \end{cases}$$

where $X_t$ is a zero-mean WSS process with PSD $S_X(f)$. We want to design a filter to help decide whether we're in case 0 or 1 above. Let the output of this system after passing through the filter $h(t)$ be $v_0(t) + Y_t$. As with our primers on estimation and detection theory, we must first decide what our objective function is. In this case, we choose to maximize the signal-to-noise ratio (SNR)

$$\begin{aligned} SNR &= \frac{v_0(t_0)^2}{\mathbb{E}\left[Y_{t_0}^2\right]} \\ &= \frac{v_0(t_0)^2}{P_Y}. \end{aligned}$$

To reiterate, $v_0(t_0)$ is the output signal after filtering in the case where an aircraft is present (case 1). To maximize the SNR, we use the trick of finding an upper bound via the Cauchy-Schwarz inequality and then designing a filter that achieves this upper bound. First examine the two terms separately.

$$P_Y = \int_{-\infty}^{\infty} S_Y(f)df = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df$$

$$\begin{aligned} v_0(t_0) &= \int_{-\infty}^{\infty} V_0(f)e^{j2\pi f t_0}df \\ &= \int_{-\infty}^{\infty} H(f)V(f)e^{j2\pi f t_0}df \\ &= \int_{-\infty}^{\infty} H(f)\sqrt{S_X(f)}\frac{V(f)}{\sqrt{S_X(f)}}e^{j2\pi f t_0}df \\ &= \int_{-\infty}^{\infty} \underbrace{H(f)\sqrt{S_X(f)}}_{g_1(f)} \underbrace{\left[\frac{V(f)^* e^{-j2\pi f t_0}}{\sqrt{S_X(f)}}\right]^*}_{g_2(f)} df, \end{aligned}$$

where $a^*$ denotes the complex conjugate of $a$. Note that the above is the functional form of an inner product between the "vectors" $g_1(f)$ and $g_2(f)$. We now apply the Cauchy-Schwarz inequality to the above to see that

$$|v_0(t)|^2 \le \underbrace{\int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df}_{P_Y} \underbrace{\int_{-\infty}^{\infty} \frac{|V(f)|^2}{S_X(f)}df}_{B}.$$

Combining the above, we see that

$$SNR = \frac{|v_0(t)|^2}{P_Y} \le \frac{P_Y B}{P_Y} = B,$$

so we want to set $h$ such that equality holds in the above statement. This occurs when the vectors $g_1(f)$ and $g_2(f)$ are aligned, i.e., when

$$H(f)\sqrt{S_X(f)} = \alpha\frac{V(f)^* e^{-j2\pi f t_0}}{\sqrt{S_X(f)}}.$$

Rearranging the above, we arrive at the transfer function for the matched filter

$$H(f) = \alpha\frac{V(f)^* e^{-j2\pi f t_0}}{S_X(f)}, \tag{1}$$

where $\alpha \in \mathbb{R}$ is set based on power/hardware considerations in practice (we typically take $\alpha$ to be the value that simplifies notation the most in this course).

**Example 2.** Let $S_X(f) = N_0/2$ (white noise). Take $\alpha = N_0/2$ as well to give

$$H(f) = V(f)^* e^{-j2\pi f t_0}.$$

In the time domain, assuming the transmitted signal $v(t)$ is real, this becomes

$$h(t) = v(t_0 - t),$$

which is a flipped version of the transmitted signal.

## 3 The Wiener Filter

The matched filter considers the detection/classification problem where we transmit a known signal. Now assume we want to estimate some unknown RP $\{V_t\}$ from a related/measured RP $\{U_t\}$. Assume that $V_t$ and $U_t$ are zero-mean, jointly WSS, and we know $S_V(f)$, $S_U(f)$, and $S_{UV}(f)$. The *Wiener filter* seeks to find the LMMSE estimate of $V_t$, i.e., to find the linear estimate $\hat{V}_t$ that minimizes $\mathbb{E}\left[\left|V_t - \hat{V}_t\right|^2\right]$. Since $\hat{V}_t$ is linear, it has the form

$$\hat{V}_t = \int_{-\infty}^{\infty} h(t-\tau)U_\tau d\tau = \int_{-\infty}^{\infty} h(\theta)U_{t-\theta}d\theta.$$

As with MMSE estimation of RVs and RVecs, we will use the orthogonality principle. In the RP/functional case, the orthogonality principle states that $h(t)$ is optimal if and only if

$$\mathbb{E}\left[\left(V_t - \hat{V}_t\right)\int_{-\infty}^{\infty} \tilde{h}(\theta)U_{t-\theta}d\theta\right] = 0$$

for every filter $\tilde{h}$. In particular, replacing $\tilde{h}$ by $h - \tilde{h}$ and letting $\tilde{V}_t$ denote the output estimate from the filter $\tilde{h}$, we see that

$$\mathbb{E}\left[\left(V_t - \hat{V}_t\right)\left(\hat{V}_t - \tilde{V}_t\right)\right] = 0.$$

We will not prove this version of the orthogonality principle, but we will use it to design $h$.

$$\begin{aligned}
0 &= \mathbb{E}\left[\left(V_t - \hat{V}_t\right)\int_{-\infty}^{\infty} \tilde{h}(\theta)U_{t-\theta}d\theta\right] \\
&= \mathbb{E}\left[\int_{-\infty}^{\infty} \tilde{h}(\theta)\left(V_t - \hat{V}_t\right)U_{t-\theta}d\theta\right] \\
&= \int_{-\infty}^{\infty} \tilde{h}(\theta)\mathbb{E}\left[\left(V_t - \hat{V}_t\right)U_{t-\theta}\right]d\theta \\
&= \int_{-\infty}^{\infty} \tilde{h}(\theta)\mathbb{E}\left[R_{VU}(\theta) - R_{\hat{V}U}(\theta)\right]d\theta.
\end{aligned}$$

Taking $\tilde{h} = R_{VU}(\theta) - R_{\hat{V}U}(\theta)$, the above becomes

$$\int_{-\infty}^{\infty} \left|R_{VU}(\theta) - R_{\hat{V}U}(\theta)\right|^2 d\theta = 0,$$

so we want to set $R_{\hat{V}U} = R_{VU}$. Since we are passing $U_t$ through an LTI system, we have

$$R_{VU}(\tau) = R_{UV}(-\tau) = \int_{-\infty}^{\infty} h(\theta)R_U(\tau-\theta)d\theta = R_{\hat{V}U}(\tau).$$

In the Fourier domain, this implies

$$S_{VU}(f) = H(f)S_U(f).$$

Solving the above for $H(f)$, we get the transfer function for the Wiener filter

$$H(f) = \frac{S_{VU}(f)}{S_U(f)}.$$

Recall our solution to LMMSE estimation of RVs was

$$\hat{X} = \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}Y.$$

Hence, the LMMSE estimator has a similar form for RPs, but this similarity appears in the Fourier domain, since "linear" operations happen via filters.

## 4 Properties of WSS RPs

We include here a number of useful properties of WSS RPs, some of which were also given in Lecture 13.

1. $R_{XY}(\tau) = R_{YX}(-\tau)$
2. $R_X(0) \geq |R_X(\tau)|$ for all $\tau$
3. $R_X(\tau)$ is periodic with period $T$ if and only if $X_t$ is periodic with period $T$
4. If $R_X(T) = R_X(0)$ for some $T$, then $R_X(\tau)$ is period with period $T$ and so is $X_t$ (with probability 1)
5. $R_X(\tau)$ is real
6. $R_X(\tau)$ is even, i.e., $R_X(\tau) = R_X(-\tau)$
7. $R_X(\tau)$ is positive semidefinite
8. $S_X(f)$ is real
9. $S_X(f)$ is even
10. $S_X(f) \geq 0$ for all $f$