EE 520: Random Processes **Fall 2021**

Lecture 8
Bivariate Random Variables

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Recommended Reading: Pishro-Nik: 5.1 - 5.3; Gubner: 7.1 - 7.5

1 Joint and Marginal Probabilities

We are often interested in functions of multiple RVs. For example, if X and Y are RVs, we may want to know the distribution of $\min(X, Y)$, XY, or X/Y. To compute such probabilities, we can use the *joint CDF*.

Definition 1. For two RVs X and Y, the **joint cumulative distribution function (CDF)** is

$$F_{XY}(x, y) = P\left(X \leq x, Y \leq y\right).$$

Note that the comma could be replaced with an intersection symbol, i.e., the above indicates the probability that $X \leq x$ *and* $Y \leq y$.

We are also interested in the probability that X and Y lie in some specified sets, i.e., that $X \in B$ and $Y \in C$.

Definition 2. The **Cartesian product** of two univariate sets $B, C \subset \mathbb{R}$ is

$$B \times C = \left\{(x, y) : x \in B, y \in C\right\}.$$

Example 1. The set $\mathbb{R} \times \mathbb{R}$ is written $\mathbb{R}^2$ and is the two-dimensional Euclidean plane, which is what we think of when we plot points in two dimensions.

Example 2. Let $B = [1, 3]$ and $C = [0, 4]$. Then $B \times C$ is the set of two-dimensional points whose first coordinate is between 1 and 3 and whose second coordinate is between 0 and 4, as depicted in blue in Fig. 1.

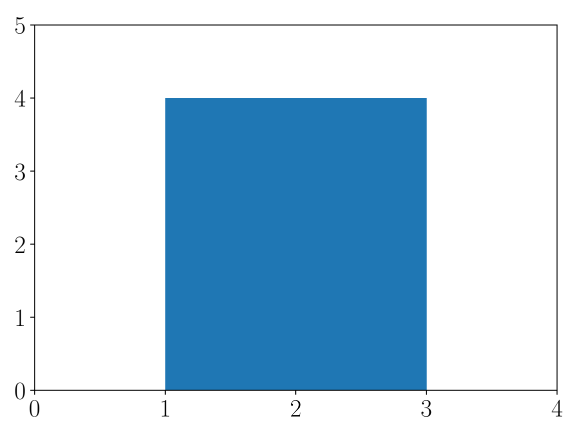

Figure 1: $B \times C$ as defined in Example 2.

For RVs, we use the shorthand notation

$$P\left(X \in B, Y \in C\right) = P\left((X,Y) \in B \times C\right),$$

so the joint CDF is equivalently written as

$$F_{XY}(x,y) = P\left((X,Y) \in (-\infty, x] \times (-\infty, y]\right).$$

Example 3. What is the probability that (X,Y) belongs to the rectangle $R = (a,b] \times (c,d]$? To solve this problem, we define a few useful sets

$$\begin{array}{rr} A = (-\infty, a] \times (-\infty, d] & B = (-\infty, b] \times (-\infty, c] \\ C = (a,b] \times (-\infty, c] & D = (-\infty, a] \times (c, d]. \end{array}$$

If we draw a picture of each of these (do this yourself!), we see that

$$R \cup (A \cup B) = (-\infty, b] \times (-\infty, d].$$

Hence,

$$F_{XY}(b,d) = P\left((X,Y) \in R\right) + P\left((X,Y) \in A \cup B\right),$$

where we have used the fact that R is disjoint from $A \cup B$. Now we can solve for $P\left((X,Y) \in A \cup B\right)$ using the inclusion-exclusion principle

$$\begin{aligned} P\left((X,Y) \in A \cup B\right) &= P\left((X,Y) \in A\right) + P\left((X,Y) \in B\right) - P\left((X,Y) \in A \cap B\right) \\ &= F_{XY}(a,d) + F_{XY}(b,c) - F_{XY}(a,c). \end{aligned}$$

Therefore

$$P\left((X,Y) \in R\right) = F_{XY}(b,d) - F_{XY}(a,d) - F_{XY}(b,c) + F_{XY}(a,c).$$

This equation is known as the *rectangle formula*.

Like with the PDF/PMF, we can also obtain the marginal CDFs from the joint CDF

$$F_X(x) = \lim_{y \to \infty} F_{XY}(x,y) \qquad F_Y(y) = \lim_{x \to \infty} F_{XY}(x,y).$$

Recall that two RVs are *independent* if and only if

$$P\left((X,Y) \in B \times C\right) = P(X \in B) P(Y \in C).$$

We may write this in terms of the CDF as well, yielding

$$F_{XY}(x,y) = F_X(x) F_Y(y).$$

2 Jointly-Continuous Random Variables

It turns out that not all pairs of continuous RVs have a joint PDF (though all have a joint CDF). However, those that do have a joint PDF get a special name.

Definition 3. Two RVs X and Y are called **jointly continuous** with joint PDF f_{XY} if

$$P\left((X,Y) \in A\right) = \int\int_A f_{XY}(x,y)\,dx\,dy.$$

For jointly-continuous RVs X and Y, we have some additional useful facts:

1. $\frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \frac{\partial^2}{\partial y \partial x} F_{XY}(x,y) = f_{XY}(x,y)$
2. $\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = f_X(x)$
3. $f_{XY}(x,y) = f_X(x) f_Y(y)$ if and only if X and Y are independent.

2.1 Conditional Probability and Expectation

When taking expectations, be sure to integrate over the joint density, i.e.,

$$\mathbb{E}\left[g(X,Y)\right] = \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} g(x,y) f_{XY}(x,y)dxdy.$$

Aside from this, all notions of conditional probability, including the Law of Total Probability and substitution law, are the same as in the univariate case. Be sure to read all definitions and examples in the text.

2.2 Two-Dimensional Transformations

Given two jointly-continuous RVs X_1 and X_2, let

$$Y_1 = g_1\left(X_1, X_2\right) \qquad Y_2 = g_2\left(X_1, X_2\right),$$

where g_1 and g_2 are differentiable and invertible functions. Then we can compute $P\left((Y_1, Y_2) \in C\right)$ using a formula we will now describe.

Definition 4. Let $g : \mathbb{R}^n \to \mathbb{R}^m$. Then the **Jacobian matrix** (or Jacobian) is the matrix of all partial derivatives, defined as

$$J = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}.$$

Example 4. Let $g_1(r,\theta) = r\cos\theta$ and $g_2(r,\theta) = r\sin\theta$. Then $g : \mathbb{R}^2 \to \mathbb{R}^2$ is the polar-Cartesian transformation

$$g(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

and the Jacobian is

$$J = \begin{bmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial \theta} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial \theta} r\sin\theta \end{bmatrix}.$$

We will be interested in the determinant of J, which is not easily defined in a single line. However, for a matrix $A \in \mathbb{R}^{2\times 2}$, the determinant has a simple formula

$$\det(A) = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}.$$

Example 5. In the polar-Cartesian transformation example above, we have

$$\det(J) = r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta \sin^2\theta\right) = r.$$

Theorem 1. Let X_1, X_2 be jointly-continuous RVs and g_1, g_2 be differentiable and invertible. Define $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$. Inverting, let $X_1 = h_1(Y_1, Y_2)$ and $X_2 = h_2(Y_1, Y_2)$. Finally, let

$$J = \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix}.$$

Then

$$P\left((Y_1, Y_2) \in C\right) = \int_C f_{X_1X_2}\left(h_1(y_1,y_2), h_2(y_1,y_2)\right)\left|\det(J)\right| dy_1dy_2,$$

and we have that

$$f_{Y_1Y_2}(y_1,y_2) = f_{X_1X_2}\left(h_1(y_1,y_2), h_2(y_1,y_2)\right)\left|\det(J)\right|,$$

where $|\det(J)|$ denotes the absolute value of the determinant of the Jacobian matrix J.

As a comparison, we saw the univariate case in Lecture 7

$$f_Y(y) = f_X(h(y)) \left| \frac{d}{dy} h(y) \right|,$$

which matches the above.