EE 520: Random Processes **Fall 2021**

Lecture 6
The Cumulative Distribution Function

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Recommended Reading: Pishro-Nik: 4.1 - 4.3, 6.2.5; Gubner: 4.4, 5.1

1 Expectation of Multiple Random Variables

For continuous RVs, functions such as the correlation and covariance have the same definition as the discrete case. For example

$$\begin{aligned} \text{cov}(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} xy f_{XY}(x,y)dydx - \int_{x=-\infty}^{\infty} x f_X(x)dx \int_{y=-\infty}^{\infty} y f_Y(y)dy. \end{aligned}$$

As stated several times in class, we often resort to bounding probabilities when we cannot compute them directly. One useful tool for doing this is *Jensen's inequality*. Jensen's inequality applies in the case where we are dealing with a *convex* function.

Definition 1. A function $g : (a,b) \to \mathbb{R}$ is called **convex** on (a,b) if

$$g(\lambda x_1 + (1-\lambda)x_2) \le \lambda g(x_1) + (1-\lambda)g(x_2)$$

for all $\lambda \in [0,1]$ and $x_1, x_2 \in (a,b)$.

Convex functions are "nice" in the sense that they only have a single minimum. An intuitive definition of a convex function is that any line segment connecting two points lies above the graph of the function.

Definition 2 (Jensen's Inequality)**.** If $g(x)$ is convex on (a,b) and $X \in (a,b)$ is a RV, then

$$g(\mathbb{E}[X]) \le \mathbb{E}[g(X)].$$

We could use direct methods to show the following, but Jensen's inequality gives it to us for free.

Example 1. Let $g(x) = |x|$, which is convex (you can see this by plotting it). Then

$$|\mathbb{E}[X]| \le \mathbb{E}[|X|].$$

2 Cumulative Distribution Functions

Definition 3. The **cumulative distribution function** (CDF) of a RV X is defined as

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)dt.$$

From the definition of the CDF, two facts are immediately obvious

1. $f_X(x) = F_X'(x)$ when F_X is differentiable
2. $P(a \le X \le b) = F_X(b) - F_X(a)$.

A few other useful properties are:

3. $\lim_{x \to \infty} F_X(x) = 1$
4. $\lim_{x \to -\infty} F_X(x) = 0$
5. $\lim_{x \searrow x_0} F_X(x) = F_X(x_0)$ (i.e., the CDF is right continuous)
6. $\lim_{x \nearrow x_0} F_X(x)$ need not equal $F_X(x_0)$.

Example 2. Let $X \sim \text{Unif}([a, b])$ (uniform distribution on the interval $[a, b]$). Then

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

This gives the CDF

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b. \end{cases}$$

We conclude with an example where $\lim_{x \nearrow x_0} \neq F_X(x_0)$.

Example 3. Let $X \sim \text{Ber}(p)$. Then

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - p & 0 \leq x \leq 1 \\ 1 & x > 1. \end{cases}$$