Ex 1

Want to know if

$$P(Y_m = j \mid Y_{m-1}, \dots, Y_0) = P(Y_m = j \mid Y_{m-1})$$

$$P(Y_m = j \mid Y_{m-1} = i_{m-1}, Y_{m-2} = i_{m-2}, \dots, Y_0 = i_0) =$$

$$P(X_{2m} = j \mid X_{2m-2} = i_{m-1}, X_{2m-4} = i_{m-2}, \dots, X_0 = i_0)$$

$$= \sum_n P(X_{2m} = j, X_{2m-1} = n \mid X_{2m-2} = i_{m-1}, \dots, X_0 = i_0)$$

$P(A, B) = P(A \mid B) P(B)$

$A = X_{2m}$

$B = X_{2m-1} \mid X_{2m-2}, \dots, X_0$

$$= \sum_n P(X_{2m} = j \mid X_{2m-1} = n, X_{2m-2} = i_{m-1}, \dots, X_0 = i_0) \times$$
$$P(X_{2m-1} = n \mid X_{2m-2} = i_{m-1}, \dots, X_0 = i_0)$$

$$= \sum_n P(X_{2m} = j \mid X_{2m-1} = n, X_{2m-2} = i_{m-1}) \times$$
$$P(X_{2m-1} = n \mid X_{2m-2} = i_{m-1})$$

$P(A \mid B, C) P(B \mid C)$
$= P(A, B \mid C)$

$$= \sum_n P(X_{2m} = j, X_{2m-1} = n \mid X_{2m-2} = i_{m-1})$$

$= P(X_{2n} = j \mid X_{2n-2} = i_{n-1})$

$= P(Y_n = j \mid Y_{n-1} = i_{n-1}) \Rightarrow Y_n$ is Markov

Ex 2

Note

$$Z_{2k+1} = \frac{Z_{2k}}{Z_{2k-1}}$$

$$P(Z_{2k+1} = 1 \mid Z_{2k} = 1, Z_{2k-1} = -1) = 0$$

$$P(Z_{2k+1} = 1 \mid Z_{2k} = 1) = \frac{1}{2}$$

$\Rightarrow Z_k$ is not Markov

Ex 3

Infinite states, so we can't write transition prob matrix. Instead use

$$\pi_j = \sum_k \pi_k P_{kj}$$

$$\pi_0 = (1-a)\pi_0 + b\pi_1 \iff \pi_1 = \frac{a}{b}\pi_0$$

$$\pi_1 = \pi_0 a + \pi_1(1-(a+b)) + \pi_2 b$$

$$\iff \pi_2 = \left(\frac{a}{b}\right)^2 \pi_0$$

In general:

$$\pi_j = \pi_{j-1} a + \pi_j (1-(a+b)) + \pi_{j+1} b$$

$$\Rightarrow \pi_j = \left(\frac{a}{b}\right)^j \pi_0$$

Also have

$$\sum_{j=0}^{\infty} \pi_j = 1 \iff \sum_{j=0}^{\infty} \left(\frac{a}{b}\right)^j \pi_0 = 1$$

$$\Rightarrow \pi_0 = 1 - \frac{a}{b}$$

$$\pi_j = \left(\frac{a}{b}\right)^j \left(1 - \frac{a}{b}\right)$$