Ex 1

a) $1 = R_x(0) < 3 = |R_x(2)|$, contradicting property (2)

b) $R_x(2) = R_x(0)$, but $R_x(\tau)$ is not periodic with period 2

c) Take $t_1 = -1$, $t_2 = 0$, $t_3 = 1$. Then

$$\det \begin{bmatrix} R_x(0) & R_x(1) & R_x(2) \\ R_x(-1) & R_x(0) & R_x(1) \\ R_x(-2) & R_x(-1) & R_x(0) \end{bmatrix} = \det \begin{bmatrix} 2 & 1.6 & 0 \\ 1.6 & 2 & 1.6 \\ 0 & 1.6 & 2 \end{bmatrix} = -2.24 \Rightarrow \text{not PSD}$$

Ex 2

We need to find

$$H(f) = \frac{S_{XY}(f)}{S_Y(f)}.$$

First note that

$$R_{XY}(\tau) = \mathbb{E}[X_t Y_{t+\tau}]$$

$$= \mathbb{E}[X_t(X_{t+\tau} + Z_{t+\tau})] = \mathbb{E}[X_t X_{t+\tau}] = R_X(\tau)$$

So

$$S_{XY}(f) = S_X(f).$$

Also,

$$R_Y(\tau) = \mathbb{E}[(X_t + Z_t)(X_{t+\tau} + Z_{t+\tau})] = \mathbb{E}[X_t X_{t+\tau}] + \mathbb{E}[Z_t Z_{t+\tau}]$$

So $S_Y(f) = S_X(f) + S_Z(f).$

Therefore

$$H(f) = \frac{P}{P+N} \quad |f| \le B$$

Ex 3

By definition, we want to find

$$H(f) = \frac{V^*(f)\, e^{-j2\pi f t_0}}{S_N(f)}$$

(let $\alpha = 1$ in the filter definition)

First take the FT of $v(t)$. From the table,

$$e^{-\lambda t} u(t) \longleftrightarrow \frac{1}{\lambda + j2\pi f},$$

So

$$V(f) = \frac{1}{c + j2\pi f} - \frac{1}{\alpha c + j2\pi f} \Rightarrow V^*(f) = \frac{(\alpha - 1)c}{(c - j2\pi f)(\alpha c - j2\pi f)}$$

Therefore

$$H(f) = \left(\frac{c^2 + (2\pi f)^2}{c}\right) \frac{e^{-j2\pi f t_0} (\alpha - 1) c}{(c - j2\pi f)(\alpha c - j2\pi f)}$$

$$= (\alpha - 1)\left(\frac{c + j2\pi f}{\alpha c - j2\pi f}\right) e^{-j2\pi f t_0}$$

Ex 4

a) Check three properties.

i) $R_x(\tau) = \cos(\tau) = \cos(-\tau) \Rightarrow R_x$ is even

ii) $|R_x(\tau)| = |\cos(\tau)| \le 1 = \cos(0) = R_x(0) \Rightarrow |R_x(\tau)| \le R_x(0)$

iii) If $\cos(\tau_0) = \cos(0)$, we have $\tau_0 = 2\pi k$ for some $k \in \mathbb{Z}$

so equality only at each period of R_x.

Note: We didn't verify that R_x is positive semi-definite, so these are not sufficient. Instead, let's inspect the PSD.

b) $S_x(f) = \frac{1}{2}\left(\delta(2\pi f - 1) + \delta(2\pi f + 1)\right)$

(note that this is real, even, and non-negative)

c) $P_x = R_x(0) = \cos(0) = 1$

d) $\mathbb{E}[X] = \mathbb{E}\left[\int_0^T u(t) X_t \, dt\right] = \int_0^T u(t)\, \mathbb{E}[X_t]\, dt = 0$

$\mathbb{E}[X^2] = \mathbb{E}\left[\int_0^T \int_0^T X_t X_s \, u(t) u(s) \, ds \, dt\right]$

$= \int_0^T \int_0^T \mathbb{E}[X_t X_s]\, u(t) u(s) \, ds \, dt$

$= \int_0^T \int_0^T \cos(s - t)\, u(t) u(s) \, ds \, dt$