Ex 1

First, we want to show that $c_1 X + c_2 Y$ is Gaussian for arbitrary $c_1$ and $c_2$. Note that

$$c_1 X + c_2 Y = c_1 X + c_2 (3X)$$

$$= (c_1 + 3c_2) X \sim N(0, (c_1 + 3c_2)^2)$$

Since $X, Y$ are zero mean, we have that $C_{XY} = \mathbb{E}[XY]$ and $C_Y = \mathbb{E}[Y^2]$.

$$\mathbb{E}[XY] = \mathbb{E}[X(3X)] = 3\,\mathbb{E}[X^2] = 3$$

$$\mathbb{E}[Y^2] = \mathbb{E}[(3X)(3X)] = 9\,\mathbb{E}[X^2] = 9$$

Therefore

$$C_{XY} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

Ex 2

Let's examine a few elements of $Y$ to look for a pattern.

$$Y_1 = X_1$$

$$Y_2 = X_1 + X_2$$

$$Y_3 = X_1 + X_2 + X_3$$

We can quickly see that $Y_n = Y_{n-1} + X_n$, or solving for $X_n$ that $X_n = Y_n - Y_{n-1}$. We can write this in matrix form as

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & & 0 \\ -1 & 1 & 0 & \cdots & & 0 \\ 0 & -1 & 1 & \cdots & & 0 \\ \vdots & & & & & \\ 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} Y_1 \\ Y_2 \\ \\ \\ Y_{n-1} \\ Y_n \end{bmatrix}$$

So $X = AY$, and $Y$ is Gaussian, so $X$ is as well.

Ex 3

First note that $Z = XV - YU$ by evaluating the determinant.

Now consider the conditional CDF

$$F_{Z|UV}(z|u,v) = P(Z \le z \mid U=u, V=v)$$

$$= P(XV - YU \le z \mid U=u, V=v)$$

$$= P(Xv - Yu \le z \mid U=u, V=v)$$

but $\begin{bmatrix} X \\ Y \end{bmatrix}$ and $\begin{bmatrix} U \\ V \end{bmatrix}$ are jointly Gaussian and uncorrelated and therefore independent, so we can drop the conditioning to see that

$$F_{Z|UV}(z|u,v) = P(Xv - Yu \le z).$$

Since $X, Y \overset{iid}{\sim} N(0,1)$, we have

$$Xv - Yu \sim N(0, u^2+v^2).$$

Therefore

$$f_{Z|U,V}(z|u,v) \sim N(0, u^2+v^2).$$