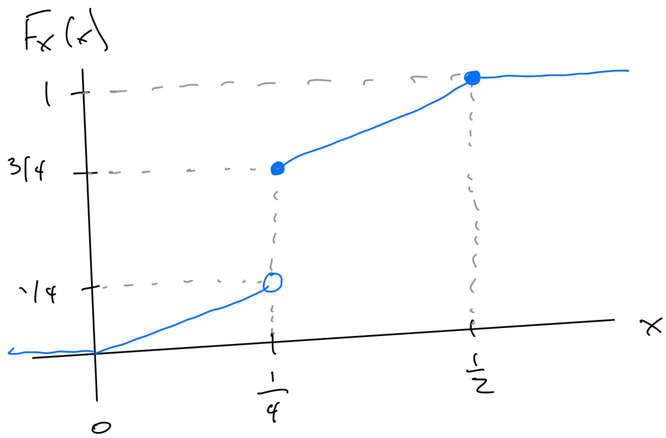
Ex 1

a) First draw the CDF of X



Note the jump at $x = \frac{1}{4}$. This can only be due to a point mass at that value, indicating X is a mixed RV.

b) $P(X \le \frac{1}{3}) = F_X(\frac{1}{3}) = (x + \frac{1}{2})\Big|_{x=\frac{1}{3}} = \frac{1}{3} + \frac{1}{2}$

c) $P(X \ge \frac{1}{4}) = 1 - P(X < \frac{1}{4})$

$$= 1 - \left(P(X \le \tfrac{1}{4}) - P(X = \tfrac{1}{4})\right)$$

$$= 1 - \left(F_X(\tfrac{1}{4}) - \tfrac{1}{2}\right)$$

$$= 1 - \tfrac{3}{4} + \tfrac{1}{2} = \tfrac{3}{4}$$

d) We can write $F_X(x) = C(x) + D(x)$, where

$$C(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le \frac{1}{2} \\ \frac{1}{2}, & x > \frac{1}{2} \end{cases}$$

$$D(x) = \begin{cases} 0, & x < \frac{1}{4} \\ \frac{1}{2}, & x \ge \frac{1}{4} \end{cases} = \frac{1}{2} u\left(x - \frac{1}{4}\right)$$

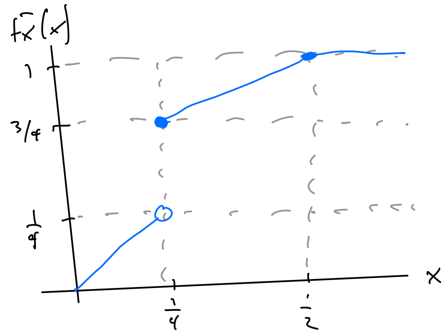
e) $$c(x) = C'(x) = \begin{cases} 0, & x < 0 \text{ or } x \ge \frac{1}{2} \\ 1, & 0 \le x < \frac{1}{2} \end{cases}$$

f) $$E[X] = \int_{-\infty}^{\infty} x\, c(x)\, dx + \sum_k x_k a_k$$

$$= \int_0^{1/2} x\, dx + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Ex 2

Find $f_X(x)$ by differentiating $F_X(x)$, paying careful attention to the jump at $x = \frac{1}{4}$:



$$f_X(x) = \begin{cases} 1, & x \in [1, \frac{1}{2}) \\ 0, & \text{else} \end{cases} + \frac{1}{2}\delta\left(x - \frac{1}{4}\right)$$

$$= \frac{1}{2}\delta\left(x - \frac{1}{4}\right) + \left(u(x) - u\left(x - \frac{1}{2}\right)\right)$$

Ex 3

Use the formula from Thm. 1. We have

$$Y = g(X) = bX + \mu \Rightarrow g^{-1}(Y) = \frac{Y - \mu}{b}$$

So $\frac{d}{dy} g^{-1}(y) = \frac{1}{b}$. Applying the formula, we have

$$f_Y(y) = \left| \frac{d}{dy} g^{-1}(y) \right| f_X\left(g^{-1}(y)\right)$$

$$= \frac{1}{b} f_X\left(\frac{Y - \mu}{b}\right) = \frac{1}{2b} \exp\left(-\frac{|y - \mu|}{b}\right)$$

which is a Laplace(μ, b) PDF.

Ex 4 :

This can either be approached using the CDF or by the formula from Thm. 1. To check that we can use Thm. 1, note that $g(x) = \sqrt{2\sigma^2 x}$ is monotone increasing for $x \geq 0$, and $X \geq 0$ since it is exponential. Now find the required terms.

$$g(x) = y = \sqrt{2\sigma^2 x} \iff x = \frac{y^2}{2\sigma^2} = g^{-1}(y)$$

$$\Rightarrow \frac{d}{dy} g^{-1}(y) = \frac{y}{\sigma^2}$$

$$f_X(g^{-1}(y)) = \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

Applying the formula,

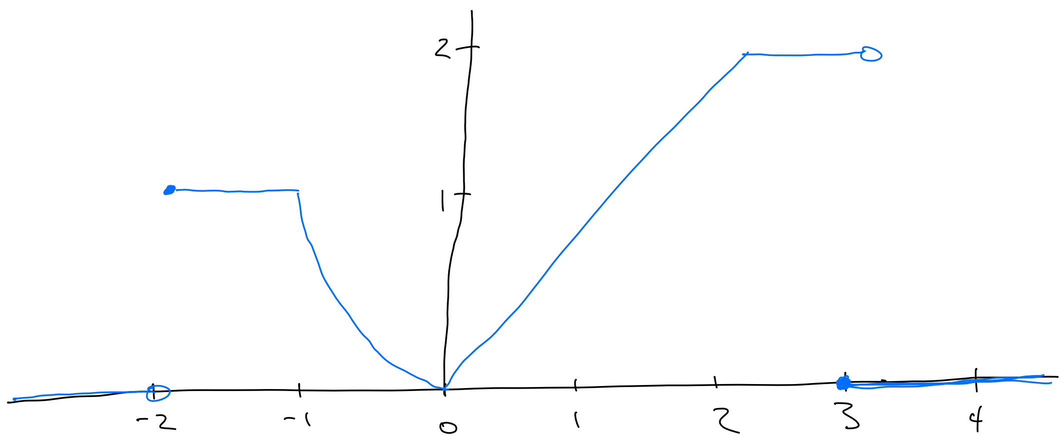
$$f_Y(y) = \left|\frac{d}{dy} g^{-1}(y)\right| f_X(g^{-1}(y))$$

$$= \frac{y}{\sigma^2} \exp\left(-\frac{y^2}{2\sigma^2}\right), \quad y \geq 0$$

which is the desired PDF.

Ex 5:

First draw a picture.



To find $F_Y(y)$, we consider all cases where something "interesting" happens.

Region 1: $y<0$

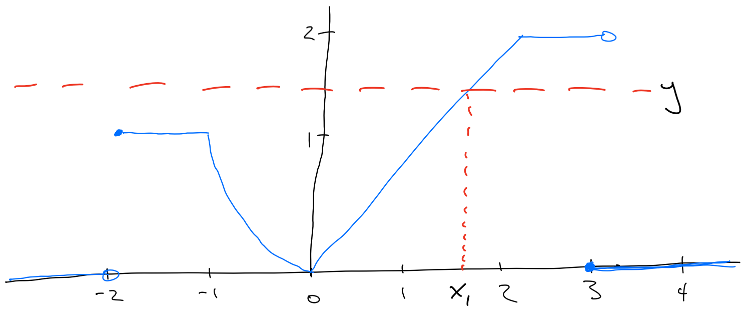
Since $0 \le Y \le 2$, we have

$$P(Y<0)=0$$

Region 2: $1 \le y < 2$

To find $P(Y \le y)$ in this region, draw a horizontal line at some $y \in [1,2)$ and examine the corresponding value(s) of x.

Note that for all $x \le x_1$ and $x > 3$, we have $g(x) \le y$.



Next, we solve for x_1 using $g(x_1)$. In the corresponding region, $g(x) = x_1$ and so $x_1 = y$. Further, for $x \le 0$, we have $g(x) \le y$. Therefore

$$P(Y \le y) = P(g(X) \le y) \qquad \{X \le y\} \cup \{X \ge 3\})$$

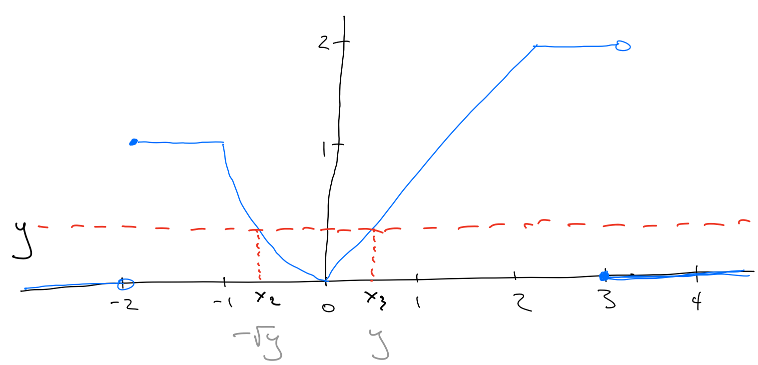
$$= P(\{X \in [-4,0]\} \cup \{X \in [0,y]\} \cup \{X \in [3,4]\})$$

$$= \frac{4}{8} + \frac{y}{8} + \frac{1}{8} \quad (\text{since } X \sim \text{Unif}([-4,4]))$$

$$= \frac{y+5}{8}$$

Region 3: $0 \le y < 1$

We again draw a horizontal line and consider the various values of x such that $g(x) \le y$.



Now we have two intersections. At x_2, we have $g(x) = x^2$, and hence $x_2 \in \{-\sqrt{y}, \sqrt{y}\}$. Note that we are in the region $[-1, 0]$, so we have $x_2 = -\sqrt{y}$. For the second intersection, we have $g(x) = x$, so $x_3 = y$. Therefore

$$g(x) \le y \quad \text{for} \quad x \in (-\sqrt{y}, y)$$

Finally, note that $g(x) \le y$ for $x < -2$ and $x \ge 3$ as well.

Therefore,

$$\begin{aligned} P(Y \le y) &= P(g(X) \le y) \\ &= P(\{X < -2\} \cup \{-\sqrt{y} < X < y\} \cup \{X \ge 3\}) \\ &= \frac{y + \sqrt{y} + 3}{8} \end{aligned}$$

<u>Region 4:</u> $y \ge 2$

Since $y \le 2$ for all x, we have

$$P(Y \le 2) = 1.$$

Putting this all together, we see that

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ (y + \sqrt{y} + 3)/8 & 0 \le y < 1 \\ (y+5)/8 & 1 \le y < 2 \\ 1 & y \ge 2 \end{cases}$$

Differentiating yields the PDF, which is given on pg. 204 of Gubner.

Ex 6:

Recall that for $Y \sim \exp(\lambda)$, $f_Y(y) = \lambda e^{-\lambda y}$. Thus

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\,dt = \int_0^y \lambda e^{-\lambda t}\,dt$$

$$= -e^{-\lambda t}\Big|_0^y = 1 - e^{-\lambda y}$$

We then take $Y = F_Y^{-1}(X)$, so we have

$$x = 1 - e^{-\lambda y} \iff 1 - x = e^{-\lambda y}$$

$$\iff y = -\frac{1}{\lambda}\ln(1-x) =: g(x).$$