Ex 1: See PN.

Ex 2:

X is equally likely to fall anywhere in the range $[2,6]$, so

$$f_X(x) = c$$

for some $c \in \mathbb{R}$. Further, we must have

$$\int_2^6 c \, dx = 1,$$

so we can find $c = \frac{1}{4}$. In this case, the CDF is

$$F_X(x) = \int_2^x f_X(x) dx = \int_2^x \frac{1}{4} dx = \frac{x-2}{4}, \quad x \in [2,6].$$

More completely,

$$F_X(x) = \begin{cases} 0, & x < 2 \\ \frac{x-2}{4}, & x \in [2,6) \\ 1, & x \geq 6 \end{cases}$$

Ex 3:

Using the definition of expectation,

$$\mathbb{E}[X^n] = \int_0^1 x^n \left(x^2 + \frac{2}{3}\right) dx$$

$$= \int_0^1 x^{n+2}\, dx + \frac{2}{3}\int_0^1 x^n\, dx$$

$$= \frac{1}{n+3} x^{n+3}\Big|_0^1 + \frac{2}{3}\frac{1}{n+1} x^{n+1}\Big|_0^1$$

$$= \frac{1}{n+3} + \frac{2}{3}\frac{1}{n+1}$$

Recall that

$$\mathrm{var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

Using the second form above, we have

$$\mathrm{var}(X) = \frac{1}{2+3} + \frac{2}{3}\frac{1}{2+1} - \left(\frac{1}{1+3} + \frac{2}{3}\frac{1}{1+1}\right)^2$$

$$= \frac{1}{5} + \frac{2}{3}\cdot\frac{1}{3} - \left(\frac{1}{4} + \frac{2}{3}\cdot\frac{1}{2}\right)^2$$

Ex 4 :

a) $F_Y(y) = P(Y \leq y) = P(e^{-X} \leq y)$

$= P(-X \leq \log(y))$

$= P(X \geq -\log(y))$

Note that the above is the complement of the CDF of X, i.e.,

$P(X \geq -\log(y)) = 1 - P(X \leq -\log(y))$

$= 1 - F_X(-\log(y))$

$= 1 - (-\log(y)) = 1 + \log(y)$

This holds for $X \in [0,1]$, or equivalently for $Y \in [\frac{1}{e}, 1]$, so

$$F_Y(y) = \begin{cases} 0, & y \leq \frac{1}{e} \\ 1 + \log(y), & y \in [\frac{1}{e}, 1] \\ 1, & y \geq 1 \end{cases}$$

b) To find the PDF, differentiate the CDF

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{1}{y}, & y \in [\frac{1}{e}, 1] \\ 0, & \text{else} \end{cases}$$

c) You can use the PDF of Y to find

$$E[Y] = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

$$= \int_{1/e}^{1} y \cdot \frac{1}{y} \, dy = \int_{1/e}^{1} dy = 1 - \frac{1}{e}$$

Alternatively, we can use the PDF of X

$$E_Y[Y] = E_X[e^{-X}] = \int_{-\infty}^{\infty} e^{-x} f_X(x) \, dx$$

$$= \int_{0}^{1} e^{-x} \cdot 1 \, dx = 1 - \frac{1}{e}$$

Ex 5:

Given that $X \sim Exp(\lambda)$, we have

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

Now note that

$$F_Y(y) = P(Y \leq y) = P(aX \leq y)$$

$$= P\left(X \leq \frac{y}{a}\right)$$

$$= F_X\left(\frac{y}{a}\right)$$

$$= \begin{cases} 0, & y < 0 \\ 1 - e^{-\frac{\lambda}{a} y}, & y \geq 0 \end{cases}$$

which is the CDF of a $Exp\left(\frac{\lambda}{a}\right)$ RV.

Ex 6 :

a) This is an example of a mixed (continuous and discrete) RV, where we have a point mass at $X=0$. Let $Z \sim Exp(2)$. Then

$$f_X(x) = 0.02\,\delta(x) + 0.98\, f_Z(z)$$

$$= 0.02\,\delta(x) + 0.98\; 2e^{-2x}\, u(x)$$

where $\delta(x)$ denotes the Dirac delta function and $u(x)$ denotes the unit step function.

b) $P(X \geq 1) = \int_1^\infty f_X(x) = \int_1^\infty 0.98 \times 2e^{-2x}\, dx = 0.98\, e^{-2}$

c) $P(X>2 \mid X\geq 1) = \dfrac{P(X>2 \wedge X\geq 1)}{P(X\geq 1)}$

$$= \frac{P(X>2)}{P(X\geq 1)} = \frac{0.98 \cdot 2e^{-4}}{0.98 \cdot 2e^{-2}} = e^{-2}$$

d) $\mathbb{E}[X] = 0.02 \times 0 + 0.98\,\mathbb{E}[Z]$, where $Z \sim Exp(2)$

$$= 0.98 \times \frac{1}{2} = 0.49$$

$$\mathbb{E}[X^2] = 0.02 \times 0 + 0.98\,\mathbb{E}[Z^2]$$

For a $Exp(\lambda)$ RV, we know

$$\mathbb{E}[Z] = \frac{1}{\lambda} \quad \text{and} \quad var(Z) = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 = \frac{1}{\lambda^2}$$

So $\mathbb{E}[Z^2] = var(Z) + (\mathbb{E}[Z])^2$

$$= \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2}$$

Therefore

$$\mathbb{E}[X^2] = 0.98 \times \frac{2}{4} = 0.49$$

and

$$var(X) = 0.49 - (0.49)^2 = 0.2499$$

Ex 7:

We'll use Jensen's inequality and the MGF. First note that e^x is convex and increasing. Hence by Jensen's inequality

$$e^{s\mathbb{E}[Y]} \le \mathbb{E}[e^{sY}] \quad \text{MGF!}$$

$$= \mathbb{E}\left[\max_i e^{sX_i}\right] \quad \text{by monotonicity of exp}$$

$$\le \sum_{i=1}^{n} \mathbb{E}[e^{sX_i}] \quad \text{Sum includes maximum, so it's at least as big (note } e^{sX_i} \ge 0)$$

$$= n e^{\frac{s^2\sigma^2}{2}} \quad \text{MGF of Gaussian}$$

Now take the log of both sides to get

$$s\mathbb{E}[Y] \le \log n + \frac{s^2\sigma^2}{2}$$

$$\Leftrightarrow \mathbb{E}[Y] \le \frac{\log n}{s} + \frac{s\sigma^2}{2}$$

This holds for any $s > 0$, so we can choose the s that gives the tightest bound. The arithmetic mean - geometric mean (AM-GM) inequality tells us that

$$\frac{\log n}{s} + \frac{s\sigma^2}{2} \ge \sqrt{\frac{\log n}{s} \frac{s\sigma^2}{2}}$$

$$= \sigma\sqrt{\frac{\log n}{2}}$$

with equality when

$$\frac{\log n}{s} = \frac{s\sigma^2}{2}.$$

Solving for s gives

$$s^2 = \frac{2\log n}{\sigma^2} \iff s = \frac{1}{\sigma}\sqrt{2\log n}$$

Plugging back in gives

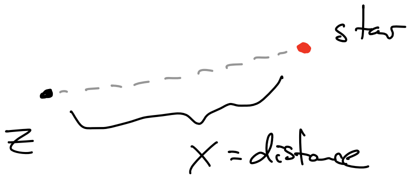
$$\mathbb{E}[Y] \leq \sigma\frac{\log n}{\sqrt{2\log n}} + \frac{\sigma}{2}\sqrt{2\log n}$$

$$= \sigma\sqrt{2\log n}$$

where we note that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Ex 8 :

Although difficult to see, this problem relies on the trick of thinking about the complement of an event. We're after the distance from a random point z to the nearest star.



Instead, let's think about the probability that there are no stars within a given distance, $P(X > x)$.

$$P(X > x) = P(\text{no stars within distance } x)$$
$$= P(N = 0 \text{ in sphere of radius } x)$$

The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$, so the corresponding PMF is $\text{Poisson}\left(\rho \frac{4}{3}\pi x^3\right)$ and

$$P(N=0) = e^{-\rho \frac{4}{3}\pi x^3}.$$

Now note that $P(X > x) = 1 - P(X \leq x) = 1 - F_X(x)$, so that

$$F_X(x) = 1 - P(X > x)$$
$$= 1 - e^{-\rho \frac{4}{3}\pi x^3}.$$

Differentiating gives

$$f_X(x) = 4\pi\rho x^2 e^{-\rho \frac{4}{3}\pi x^3}$$

Ex 9

You are served by teller 1 if $X_1 < X_2$, so we are after

$$P(X_1 < X_2).$$

Since X_1 and X_2 are independent, we have that

$$f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$
$$= \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2}.$$

Therefore

$$P(X_1 < X_2) = \int_{x_2=0}^{\infty} \int_{x_1=0}^{x_2} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2}\, dx_1\, dx_2$$

$$= \int_{x_1=0}^{\infty} \int_{x_2=x_1}^{\infty} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2}\, dx_2\, dx_1$$

Both are true, but the second is easier to evaluate.

$$= \int_{x_1=0}^{\infty} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 \int_{x_2=x_1}^{\infty} e^{-\lambda_2 x_2}\, dx_2\, dx_1$$

$$= \int_{x_1=0}^{\infty} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 \underbrace{\left(-\frac{1}{\lambda_2} e^{-\lambda_2 x_2} \Big|_{x_2=x_1}^{\infty} \right)}_{\frac{1}{\lambda_2} e^{-\lambda_2 x_1}} dx_1$$

$$= \int_{x_1=0}^{\infty} \lambda_1 e^{-\lambda_1 x_1} e^{-\lambda_2 x_1} dx_1$$

$$= \int_{x_1=0}^{\infty} \lambda_1 e^{-x_1(\lambda_1+\lambda_2)} dx_1$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2}$$

Note that the probability does not depend on your arrival time. This implies that the exponential distribution is memoryless.