Ex 1: See PN, 5.1.5, Example 5.12.

Ex 2: See PN, 5.1.5, Example 5.13 and below text.

Ex 3: See PN, 5.1.5, Example 5.14.

Ex 4: See PN, 5.3.1, Example 5.32.

Ex 5: See PN.

Ex 6: See PN, 6.2.4, Example 6.23.

Ex 7:

Thinking directly about the event $E = \{X > 1\}$ would require computation in the case of $X = 2, 3, 4, \ldots$, which may be difficult. However, $E^c = \{X = 0 \cup X = 1\}$, which seems easier.

$$P(X>1) = 1 - P(X \leq 1)$$

$$= 1 - \left(P(X=0) + P(X=1)\right)$$

$$= 1 - \left(e^{-\lambda} + \lambda e^{-\lambda}\right) = 1 - e^{-\lambda}(1+\lambda).$$

If we had the CDF handy, we could simply write

$$P(X>1) = 1 - P(X \leq 1)$$

$$= 1 - F_X(1)$$

$$= e^{-\lambda} \sum_{i=0}^{1} \frac{\lambda^i}{i!}$$

$$= e^{-\lambda}(1+\lambda).$$

Ex 8:

Consider the k^{th} student. We want the probability they get an "A" AND that nobody else gets an "A." Let

$$X_i = \begin{cases} 1 & i^{th} \text{ student gets an "A"} \\ 0 & \text{otherwise.} \end{cases}$$

For student k, we're interested in

$$E_k = \{ X_k = 1 \wedge X_\ell = 0,\ \ell \neq k \}$$

By independence, since $P(X_i) = p$ for all i,

$$P(E_k) = p(1-p)^{14}.$$

Now we need to allow for $k=1$ OR $k=2$ OR ..., so the probability of interest is

$$P(E) = P\left(\bigcup_{k=1}^{15} P(E_k) \right) = \sum_{k=1}^{15} P(E_k) = 15p(1-p)^{14}$$

disjoint events

Ex 9:

From the problem, we have

$$P_N(n) = \frac{\lambda^n}{n!} e^{-\lambda} \quad \text{and} \quad P_{K|N}(k|n) = \binom{n}{k} p^k (1-p)^{n-k}$$

Poisson

binomial

Therefore

for a specific n

mean of binomial distribution

$$\mathbb{E}[K \mid N = n] = pn$$

and for a random N, we have

$$\mathbb{E}[K \mid N] = pN.$$

We can find $\mathbb{E}[K]$ using LTE, version 2

$$\mathbb{E}[K] = \mathbb{E}_N\left[\mathbb{E}_{K|N}[K \mid N]\right]$$

$$= \mathbb{E}_N[pN] = p\lambda$$

To find $\mathbb{E}[N|K]$, we can use Bayes' rule

$$p_{N|K}(n|k) = \frac{p_{K|N}(k|n)\, p_N(n)}{\sum_{m \geq k} p_{K|N}(k|m)\, p_N(m)} \quad \text{if } n \geq k$$

$$= \frac{\binom{n}{k} p^k (1-p)^{n-k} \left(\lambda^n / n!\right) e^{-\lambda}}{\sum_{m \geq k} \binom{m}{k} p^k (1-p)^{m-k} \left(\lambda^m / m!\right) e^{-\lambda}}$$

use $\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{\lambda}$ and algebra

$$= \frac{\left((1-p)\lambda\right)^{n-k}}{(n-k)!} e^{-\lambda(1-p)}$$

From this, we can compute

$$\mathbb{E}[N|K=k] = \sum_{n \geq k} n\, p_{N|K}(n|k) = k + q\lambda$$

where $q = 1-p$, which gives the final result

$$\mathbb{E}[N|K] = K + q\lambda.$$

Ex 10 :

We have M people and can envision each as randomly selecting a floor to exit. Let X be the number of stops made. Then we can again use the LTE (version 2) to see that

$$\mathbb{E}[X] = \mathbb{E}_M\left[\mathbb{E}_{X|M}[X|M]\right]$$

First fix $M=m$. It is easier to talk about the number of floor where no stop is made, so call $Y = n - X$ that number.

Now introduce the indicator RVs

$$A_i = \begin{cases} 1 & \text{no stop on floor } i \\ 0 & \text{otherwise} \end{cases}$$

so that $Y = \sum_{i=1}^{n} A_i$. We have that

$$P(A_i = 1) = \underbrace{\left(1 - \tfrac{1}{n}\right)^m}_{}$$

m experiments where everyone chooses from among $n-1$ floors of n total

Therefore

$$\mathbb{E}[Y] = \sum_{i=1}^{n} \left(1 - \tfrac{1}{n}\right)^m = n\left(1 - \tfrac{1}{n}\right)^m$$

and

$$\mathbb{E}[X \mid M=m] = n - \mathbb{E}[Y] = n - n\left(1 - \tfrac{1}{n}\right)^m$$

We are now ready to find $\mathbb{E}[X]$.

$$\mathbb{E}[X] = \mathbb{E}_M\left[\mathbb{E}_{X|M}[X|M]\right]$$

$$= \sum_{m=0}^{\infty} \mathbb{E}_{X|M}[X|M=m]\, p_M(m)$$

$$= \sum_{m=0}^{\infty} \left(n - n\left(1-\tfrac{1}{n}\right)^m\right) p_M(m)$$

$$= n \sum_{m=0}^{\infty} p_M(m) - n \sum_{m=0}^{\infty} \left(1-\tfrac{1}{n}\right)^m p_M(m)$$

summing over the entire PMF gives 1

$$= n - n \sum_{m=0}^{\infty} \left(1-\tfrac{1}{n}\right)^m \frac{e^{-10}\, 10^m}{m!}$$

$$= n - n e^{-10} \sum_{m=0}^{\infty} \frac{\left(10\left(1-\tfrac{1}{n}\right)\right)^m}{m!}$$

Now use the important fact that $\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{\lambda}$ to see that

$$\mathbb{E}[X] = n - n e^{-10} \exp\left(10\left(1-\tfrac{1}{n}\right)\right)$$

$$= n\left(1 - e^{-10/n}\right)$$